Tutorial Notes 5

1. Evaluate
$$\int_C (xy + y + z)\,\mathrm{d}s,$$
where C: $(2t, t, 2-2t)$, $0 \le t \le 1$.

Solutions:

Let $\gamma(t) = (2t, t, 2-2t)$. Then $\dot{\gamma} = (2, 1, -2)$ and $|\dot{\gamma}(t)| = 3$. The integral is equal to
$$\int_0^1 (2t \cdot t + t + 2 - 2t) \cdot 3\,\mathrm{d}t = \frac{13}{2}.$$

2. Evaluate
$$\int_C (-\sqrt{x^2 + z^2})\,\mathrm{d}s,$$
where C: $(0, a\cos t, a\sin t)$, $0 \le t \le 2\pi$.

Solutions:

Let $\gamma(t) = (0, a\cos t, a\sin t)$. Then $\dot{\gamma}(t) = (0, -a\sin t, a\cos t)$ and $|\dot{\gamma}(t)| = a$. The integral is equal to
$$\int_0^{2\pi} (-a|\sin t|) \cdot a\,\mathrm{d}t = -2a^2 \int_0^{\pi} \sin t\,\mathrm{d}t = -4a^2.$$

3. Find the area of the "winding wall" standing orthogonally on the curve $y = x^2$, $0 \le x \le 2$, and beneath the surface $f(x, y) = x + \sqrt{y}$.

Solutions:

The area is
$$\int_C f(x, y)\,\mathrm{d}s,$$
where C: $y = x^2$, $0 \le x \le 2$. A parametrization of the curve is $\gamma(x) = (x, x^2)$. Then $\dot{\gamma}(x) = (1, 2x)$ and $|\dot{\gamma}(x)| = \sqrt{1 + 4x^2}$. Hence the integral is
$$\int_0^2 (x + \sqrt{x^2}) \cdot \sqrt{1 + 4x^2}\,\mathrm{d}x = \int_0^2 \sqrt{1 + 4x^2}\,\mathrm{d}(x^2) = \int_0^4 \sqrt{1 + 4u}\,\mathrm{d}u = \frac{17\sqrt{17} - 1}{6}.$$

4. Find the work done by the gradient of $f(x, y) = (x + y)^2$ counterclockwise around the circle $x^2 + y^2 = 4$ from $(2, 0)$ to itself.

Solutions:

$\nabla f(x, y) = (2(x + y), 2(x + y))$. Use the parametrization $\gamma(t) = (2\cos t, 2\sin t)$,

$0 \leq t \leq 2\pi$. Then $\dot{\gamma}(t) = (-2\sin t, 2\cos t)$. The work is

$$\begin{aligned}
&\int_0^{2\pi} [2(2\cos t + 2\sin t)(-2\sin t) + 2(2\cos t + 2\sin t)(2\cos t)]\,\mathrm{d}t \\
= 8 &\int_0^{2\pi} (\cos^2 t - \sin^2 t)\,\mathrm{d}t = 0.
\end{aligned}$$

Remark 1

If the force F is the gradient of a function ϕ (the force is called conservative), the work of it around a closed curve C is 0. Indeed, take a parametrization of the curve: $\gamma(t)$, $0 \leq t \leq 1$, then the work is

$$\int_0^1 \nabla\phi(\gamma(t))\dot{\gamma}(t)\,\mathrm{d}t = \int_0^1 \phi(\gamma(t))'\,\mathrm{d}t = 0.$$

5. Find the circulation and flux of the vector fields

$$F_1 = (x, y), \quad F_2 = (-y, x)$$

around and across the following curves:

(a) the circle $(\cos t, \sin t)$, $0 \leq t \leq 2\pi$;

(b) the ellipse $(\cos t, 4\sin t)$, $0 \leq t \leq 2\pi$.

Solutions:

We use $C(X, L)$ and $F(X, L)$ to denote the circulation and flux of the vector field X around and across the curve L. Moreover, we let $(u, v)^\perp = (-v, u)$. Then we note that $F_2 = F_1^\perp$. It follows that

$$C(F_2, L) = F(F_1, L) \quad \text{and} \quad F(F_2, L) = -C(F_1, L).$$

Indeed,

$$\begin{aligned}
C(F_2, L) &= \int_L F_2 \cdot T\,\mathrm{d}s = \int_L F_1^\perp \cdot T\,\mathrm{d}s = -\int_L F_1 \cdot T^\perp\,\mathrm{d}s \\
&= \int_L F_1 \cdot n\,\mathrm{d}s = F(F_1, L).
\end{aligned}$$

Similarly,

$$\begin{aligned}
F(F_2, L) &= \int_L F_2 \cdot n\,\mathrm{d}s = \int_L F_1^\perp \cdot n\,\mathrm{d}s = -\int_L F_1 \cdot n^\perp\,\mathrm{d}s \\
&= -\int_L F_1 \cdot T\,\mathrm{d}s = -C(F_1, L).
\end{aligned}$$

Hence it suffices to calculate the circulation and flux of F_1. Denote the circle and ellipse by L_a and L_b.

(a)

$$C(F_1, L_a) = \int_{L_a} F_1 \cdot T \,\mathrm{d}s = \int_{L_a} n \cdot T \,\mathrm{d}s = 0.$$

$$F(F_1, L_a) = \int_{L_a} F_1 \cdot n \,\mathrm{d}s = \int_{L_a} n \cdot n \,\mathrm{d}s = 2\pi.$$

(b)

$$C(F_1, L_b) = \int_{L_b} (x \,\mathrm{d}x + y \,\mathrm{d}y) = \int_0^{2\pi} [\cos t(-\sin t) + 4\sin t(4\cos t)] \,\mathrm{d}t = 0.$$

$$F(F_1, L_b) = \int_{L_b} (x \,\mathrm{d}y - y \,\mathrm{d}x) = \int_0^{2\pi} [\cos t(4\cos t) - 4\sin t(-\sin t)] \,\mathrm{d}t = 8\pi.$$