Tutorial Notes 11

1. S is the surface $y = \log x$, $1 \leq x \leq \mathrm{e}$, $0 \leq z \leq 1$. Let the unit normal vector n point away from the xz-plane. Find the flux of $F = (0, 2y, z)$ across S.

   Solutions:

   $$\mathrm{d}S = \sqrt{1 + \frac{1}{x^2}}\,\mathrm{d}x\,\mathrm{d}z.$$

   and

   $$n = \frac{\left(-\frac{1}{x}, 1, 0\right)}{\sqrt{\frac{1}{x^2} + 1}}.$$

   Then the flux is

   $$\int_0^1 \int_1^{\mathrm{e}} 2y\,\mathrm{d}x\,\mathrm{d}z = \int_1^{\mathrm{e}} 2\log x\,\mathrm{d}x = 2.$$

2. Find the circulation of $F = (2y, 3x, -z^2)$ around the circle C: $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above.

   Solutions:

   By Stokes' theorem,

   $$\int_C F \cdot \mathrm{d}r = \int_S (\nabla \times F) \cdot n\,\mathrm{d}S,$$

   where S: $z = 0$, $x^2 + y^2 \leq 9$, with $n = (0, 0, 1)$. $(\nabla \times F)_3 = 1$. Hence the flux is 9π.

3. Let S: $4x^2 + y + z^2 = 4$, $y \geq 0$, with the outer normal vector and let

   $$F = \left(-z + \frac{1}{2 + x}, \arctan y, x + \frac{1}{4 + z}\right).$$

   Find

   $$\int_S (\nabla \times F) \cdot n\,\mathrm{d}S.$$

   Solutions:

   Applying Stokes' theorem twice, we have

   $$\int_S (\nabla \times F) \cdot n\,\mathrm{d}S = \int_C F \cdot \mathrm{d}r = \int_{S_0} (\nabla \times F) \cdot n\,\mathrm{d}S,$$

   where C: $4x^2 + z^2 = 4$, $y = 0$, clockwise when viewed from $y > 0$ and S_0: $y = 0$, $4x^2 + z^2 \leq 4$, with the unit normal vector $(0, 1, 0)$. $(\nabla \times F)_2 = -2$. Hence the integral is -4π.

4. Let $F = (2z, 3x, 5y)$ and S: $(r\cos\theta, r\sin\theta, 4 - r^2)$, $0 \le r \le 2$, $0 \le \theta \le 2\pi$, with the outer normal vector. Find the flux of F across S.

   Solutions:

   Applying Stokes' theorem twice, we have

$$\int_S (\nabla \times F) \cdot n \, \mathrm{d}S = \int_C F \cdot \mathrm{d}r = \int_{S_0} (\nabla \times F) \cdot n \, \mathrm{d}S,$$

   where C: $x^2 + y^2 = 4$, $z = 0$, counterclockwise when viewed form above and S_0: $z = 0$, $x^2 + y^2 \le 4$, with the unit normal vector $(0, 0, 1)$. $(\nabla \times F)_3 = 3$. Hence the flux is 12π.

5. Let C be a simple closed smooth curve in the plane $2x + 2y + z = 2$ counterclockwise when viewed from $y > 0$. Show that

$$\int_C 2y \, \mathrm{d}x + 3z \, \mathrm{d}y - x \, \mathrm{d}z$$

   depends only on the area of the region enclosed by C.

   Solutions:

   Suppose that S is enclosed by C with the normal vector $(2, 2, 1)$. It follows from Stokes' theorem that

$$\begin{aligned}
\int_C 2y \, \mathrm{d}x + 3z \, \mathrm{d}y - x \, \mathrm{d}z &= \int_S [\nabla \times (2y, 3z, -x)] \cdot n \, \mathrm{d}S \\
&= \int_S (-3, 1, -2) \cdot \frac{(2, 2, 1)}{3} \, \mathrm{d}S \\
&= -2|S|.
\end{aligned}$$