MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 9

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.6

37. Find the flux of the field $\mathbf{F}(x,y,z) = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Solution. The surface has a parametrization

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4-y^2)\mathbf{k}, \qquad (x,y) \in \Omega := \{(x,y) : 0 \le x \le 1,\ -2 \le y \le 2\}.$$

Then

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k} \qquad \text{(outward normal)},$$

and

$$\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = \big((4-y^2)^2\mathbf{i} + x\mathbf{j} - 3(4-y^2)\mathbf{k}\big) \cdot (2y\mathbf{j} + \mathbf{k}) = 2xy - 3(4-y^2).$$

Hence, the flux through the surface is

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_\Omega \mathbf{F}(\mathbf{r}(x,y)) \cdot \frac{\mathbf{r}_x \times \mathbf{r}_y}{|\mathbf{r}_x \times \mathbf{r}_y|} |\mathbf{r}_x \times \mathbf{r}_y|\, dx\, dy \\ &= \int_0^1 \int_{-2}^2 \left[2xy - 3(4-y^2)\right]\, dy\, dx \\ &= \int_0^1 -32\, dx = -32. \end{aligned}$$

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42. Find the outward flux of the field $\mathbf{F} = xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}$ across the surface of the upper cap cut from the solid sphere $x^2 + y^2 + z^2 \le 25$ by the plane $z = 3$.

Solution. Across the cap S_1: The surface is given by the level surface

$$g(x,y,z) := x^2 + y^2 + z^2 = 25, \qquad (x,y) \in \Omega := \{(x,y) : x^2 + y^2 \le 16\}.$$

Then an outward unit normal is

$$\mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{2x\mathbf{i} + 2y\mathbf{j} + 2z\mathbf{k}}{10} = \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{5},$$

and

$$\mathbf{F} \cdot \mathbf{n} = (xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}) \cdot \frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{5} = \frac{x^2z + y^2z + z}{5}.$$

Since $z > 0$ on the surface, $\frac{\partial g}{\partial z} = 2z \neq 0$, and so

$$d\sigma = \frac{|\nabla g|}{|\frac{\partial g}{\partial z}|}\, dx\, dy = \frac{\sqrt{4x^2 + 4y^2 + 4z^2}}{|2z|}\, dx\, dy = \frac{5}{z}\, dx\, dy.$$

Hence, the outward flux through the surface S_1 is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_\Omega \frac{x^2 z + y^2 z + z}{5} \cdot \frac{5}{z}\, dx\, dy = \iint_\Omega (x^2 + y^2 + 1)\, dx\, dy$$
$$= \int_0^{2\pi} \int_0^4 (r^2 + 1)\, r dr\, d\theta = 144\pi.$$

Across the bottom S_2: The surface is given by the level surface

$$h(x, y, z) \coloneqq z = 3, \qquad (x, y) \in \Omega \coloneqq \{(x, y) : x^2 + y^2 \leq 16\}.$$

Then an outward unit normal is $-\mathbf{k}$ and

$$\mathbf{F} \cdot \mathbf{n} = (xz\mathbf{i} + yz\mathbf{j} + \mathbf{k}) \cdot (-\mathbf{k}) = -1.$$

Since $\frac{\partial h}{\partial z} = 1 \neq 0$, $d\sigma = dx\, dy$. Hence, the outward flux through the surface S_2 is

$$\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_\Omega -1\, dx\, dy = -\pi(4)^2 = -16\pi.$$

Therefore, the total outward flux is $144\pi - 16\pi = 128\pi$. ◀

Exercises 16.7

3. Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

   $\mathbf{F} = y\mathbf{i} + xz\mathbf{j} + x^2\mathbf{k}$

   C: The boundary of the triangle cut from the plane $x + y + z = 1$ by the first octant, counterclockwise when viewed from above.

   Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & xz & x^2 \end{vmatrix} = -x\mathbf{i} - 2x\mathbf{j} + (z - 1)\mathbf{k}.$$

   and the plane $g(x, y, z) \coloneqq x + y + z = 1$ has unit normal $\mathbf{n} = \frac{1}{\sqrt{3}}(\mathbf{i} + \mathbf{j} + \mathbf{k})$. Then

$$\nabla \times \mathbf{F} \cdot \mathbf{n} = \frac{1}{\sqrt{3}}(-3x + z - 1),$$

   and $d\sigma = \frac{|\nabla g|}{|\frac{\partial g}{\partial z}|}\, dx\, dy = \sqrt{3}\, dx\, dy$. By Stokes' Theorem, the circulation of the field $\mathbf{F}$ around the curve C is

$$\oint_C \mathbf{F} \cdot\, d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \int_0^1 \int_0^{1-x} [-3x + (1 - x - y) - 1]\, dy\, dx$$
$$= \int_0^1 \int_0^{1-x} (-4x - y)\, dy\, dx = -\frac{5}{6}.$$

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6. Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

$\mathbf{F} = x^2y^3\mathbf{i} + \mathbf{j} + x\mathbf{k}$

C: The intersection of the cylinder $x^2 + y^2 = 4$ and the hemisphere $x^2 + y^2 + z^2 = 16$, $z \geq 0$, counterclockwise when viewed from above.

Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^3 & 1 & x \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} - 3x^2y^2\mathbf{k} = -3x^2y^2\mathbf{k},$$

and C is the boundary of the disk $S := \{(x, y, 2\sqrt{3}) : x^2 + y^2 \leq 4\}$ with upward unit normal $\mathbf{n} = \mathbf{k}$.

By Stokes' Theorem, the circulation of the field $\mathbf{F}$ around the curve C is

$$\begin{aligned} \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{\{x^2+y^2 \leq 4\}} -3x^2y^2\, dx\, dy \\ &= -3 \int_0^{2\pi} \int_0^2 (r\cos\theta)^2(r\sin\theta)^2\, r dr\, d\theta = -\frac{3}{4} \int_0^{2\pi} \sin^2 2\theta\, d\theta \int_0^2 r^5\, dr \\ &= -\frac{3}{8}(2\pi)\frac{2^6}{6} = -8\pi. \end{aligned}$$

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9. Let S be the cylinder $x^2 + y^2 = a^2$, $0 \leq z \leq h$, together with its top, $x^2 + y^2 \leq a^2$, $z = h$. Let $\mathbf{F} = -y\mathbf{i} + x\mathbf{j} + x^2\mathbf{k}$. Use Stoke's Theorem to find the flux of $\nabla \times \mathbf{F}$ outward through S.

Solution. The surface S has a boundary $C : x^2 + y^2 = a^2, z = 0$ which has a parametrization

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \qquad 0 \leq t \leq 2\pi.$$

Then $\mathbf{r}'(t) = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$ and

$$\mathbf{F} \cdot \mathbf{r}'(t) = (-a\sin t)(-a\sin t) + (a\cos t)(a\cos t) = a^2.$$

By Stokes' Theorem, the flux of $\nabla \times \mathbf{F}$ outward through S is

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \oint_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} a^2 = 2\pi a^2.$$

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12. Suppose $\mathbf{F} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = (y + \sqrt{z})\mathbf{i} + e^{xyz}\mathbf{j} + \cos(xz)\mathbf{k}.$$

Determine the flux of $\mathbf{F}$ outward through the whole unit sphere $x^2 + y^2 + z^2 = 1$.

Solution. If A is C^1, then Stokes' Theorem implies that the outward flux is 0 since the whole unit sphere has no boundary. However, A (and hence F) is not defined when $z < 0$, so there is probably a problem in the question. ◀

14. Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

$\mathbf{F} = (y-z)\mathbf{i} + (z-x)\mathbf{j} + (x+z)\mathbf{k}$

$S: \quad \mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + (9-r^2)\mathbf{k}, \quad 0 \le r \le 3, \quad 0 \le \theta \le 2\pi.$

Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y-z & z-x & x+z \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 2\mathbf{k},$$

and

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} + r\mathbf{k}.$$

Hence,

$$\begin{aligned} \text{The flux of } \nabla \times \mathbf{F} \text{ across } S &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^3 (-2r^2\cos\theta - 4r^2\sin\theta - 2r)\, dr\, d\theta \\ &= -18\pi. \end{aligned}$$

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17. Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

$\mathbf{F} = 3y\mathbf{i} + (5-2x)\mathbf{j} + (x^2-2)\mathbf{k}$

$S: \ \mathbf{r}(\phi,\theta) = (\sqrt{3}\sin\phi\cos\theta)\mathbf{i} + (\sqrt{3}\sin\phi\sin\theta)\mathbf{j} + (\sqrt{3}\cos\phi)\mathbf{k}, \quad 0 \le \phi \le \pi/2, \ 0 \le \theta \le 2\pi.$

Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & (5-2x) & (x^2-2) \end{vmatrix} = -5\mathbf{k},$$

$$\begin{aligned} \mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \sqrt{3}\cos\phi\cos\theta & \sqrt{3}\cos\phi\sin\theta & -\sqrt{3}\sin\phi \\ -\sqrt{3}\sin\phi\sin\theta & \sqrt{3}\sin\phi\cos\theta & 0 \end{vmatrix} \\ &= (3\sin^2\phi\cos\theta)\mathbf{i} + (3\sin^2\phi\sin\theta)\mathbf{j} + (3\sin\phi\cos\phi)\mathbf{k}. \end{aligned}$$

Hence, the flux of $\nabla \times \mathbf{F}$ across S is

$$\begin{aligned} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_R (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_\phi \times \mathbf{r}_\theta)\, d\phi\, d\theta \\ &= \int_0^{2\pi} \int_0^{\pi/2} (-15\cos\phi\sin\phi)\, d\phi\, d\theta \\ &= -15\pi. \end{aligned}$$

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22. **Zero circulation** Let $f(x, y, z) = (x^2+y^2+z^2)^{-1/2}$. Show that the clockwise circulation of the field $\mathbf{F} = \nabla f$ around the circle $x^2 + y^2 = a^2$ in the xy-plane is zero.

(a) by taking $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le 2\pi$, and integrating $\mathbf{F} \cdot d\mathbf{r}$ over the circle.

(b) by applying Stokes' Theorem.

Solution. (a) Note that

$$\mathbf{F} = \nabla f = -\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}},$$

and

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\frac{(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}}{(a^2)^{3/2}} \cdot ((-a\sin t)bi + (a\cos t)\mathbf{j}) = 0.$$

Hence,

$$\text{Circulation} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F} \cdot \frac{d\mathbf{r}}{dt}\, dt = 0.$$

(b) Note that $\nabla \times \mathbf{F} = \nabla \times \nabla f = \mathbf{0}$. Hence, by Stokes' Theorem,

$$\text{Circulation} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_S 0\, d\sigma = 0.$$

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25. Find a vector field with twice-differentiable components whose curl is $x\mathbf{i}+y\mathbf{j}+z\mathbf{k}$ or prove that no such field exists.

Solution. Suppose $\mathbf{F}$ is a field such that $\nabla \times \mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$. Then

$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

but

$$\nabla \cdot (x\mathbf{i} + y\mathbf{j} + z\mathbf{k}) = 1 + 1 + 1 = 3.$$

Contradiction. So no such field $\mathbf{F}$ exists. ◀