# MATH 2020B Advanced Calculus II
# 2023-24 Term 2
# Suggested Solution of Homework 8

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

## Exercises 16.5

5. Find a parametrization of the surface.

   **Spherical cap** The cap cut from the sphere $x^2+y^2+z^2=9$ by the cone $z=\sqrt{x^2+y^2}$.

   **Solution.** In cylindrical coordinates, let $x=r\cos\theta$, $y=r\sin\theta$, then $z=\sqrt{9-r^2}$, $z\geq 0$. For the domain of $r$: $z=\sqrt{x^2+y^2}$ and $x^2+y^2+z^2=9$ imply

   $$2r^2=9 \implies r=\frac{3}{\sqrt{2}} \implies 0\leq r\leq \frac{3}{\sqrt{2}}$$

   Hence, a parametrization is

   $$\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+\sqrt{9-r^2}\mathbf{k}, \qquad 0\leq r\leq\frac{3}{\sqrt{2}},\ 0\leq\theta\leq 2\pi.$$

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9. Find a parametrization of the surface.

   **Parabolic cylinder between planes** The surface cut from the parabolic cylinder $z=4-y^2$ by the planes $x=0$, $x=2$, and $z=0$.

   **Solution.** When $z=0$, $0=z=4-y^2 \implies y=\pm 2$. Hence, aparametrization is

   $$\mathbf{r}(x,y)=x\mathbf{i}+y\mathbf{j}+(4-y^2)\mathbf{k}, \qquad 0\leq x\leq 2,\ -2\leq y\leq 2.$$

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18. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

    **Plane insider cylinder** The portion of the plane $z=-x$ inside the cylinder $x^2+y^2=4$.

    **Solution.** A parametrization is

    $$\mathbf{r}(r,\theta)=(r\cos\theta)\mathbf{i}+(r\sin\theta)\mathbf{j}+(-r\cos\theta)\mathbf{k}, \qquad 0\leq r\leq 2,\ 0\leq\theta\leq 2\pi.$$

    Then $\mathbf{r}_r=(\cos\theta)\mathbf{i}+(\sin\theta)\mathbf{j}-(\cos\theta)\mathbf{k}$ and $\mathbf{r}_\theta=(-r\sin\theta)\mathbf{i}+(r\cos\theta)\mathbf{j}+(r\sin\theta)\mathbf{k}$, and thus

    $$\mathbf{r}_r\times\mathbf{r}_\theta=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ \cos\theta & \sin\theta & -\cos\theta\\ -r\sin\theta & r\cos\theta & r\sin\theta\end{vmatrix}=r\mathbf{i}+r\mathbf{k},$$

    $$|\mathbf{r}_r\times\mathbf{r}_\theta|=\sqrt{r^2+r^2}=r\sqrt{2}.$$

    Hence,

    $$\text{Surface area}=\int_0^{2\pi}\int_0^2 r\sqrt{2}\,dr\,d\theta.=4\pi\sqrt{2}.$$

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34. **Hyperbolic of one sheet**

(a) Find a parametrization for the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$ in terms of the angle $\theta$ associated with the circle $x^2 + y^2 = r^2$ and the hyperbolic parameter $u$ associated with the hyperbolic function $r^2 - z^2 = 1$. (*Hint*: $\cosh^2 u - \sinh^2 u = 1$.)

(b) Generalize the result in part (a) to the hyperboloid $(x^2/a^2) + (y^2/b^2) - (z^2/c^2) = 1$.

**Solution.** (a) A parametrization is

$$\mathbf{r}(\theta, u) = (\cosh u \cos\theta)\mathbf{i} + (\cosh u \sin\theta)\mathbf{j} + (\sinh u)\mathbf{k}, \quad u \in \mathbb{R},\ \theta \in [0, 2\pi].$$

(b) A parametrization is

$$\mathbf{r}(\theta, u) = (a\cosh u \cos\theta)\mathbf{i} + (b\cosh u \sin\theta)\mathbf{j} + (c\sinh u)\mathbf{k}, \quad u \in \mathbb{R},\ \theta \in [0, 2\pi].$$

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35. *(Continuation of Exercise 34.)* Find a Cartesian equation for the plane tangent to the hyperboloid $x^2 + y^2 - z^2 = 25$ at the point $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$.

**Solution.** Consider the parametrization

$$\mathbf{r}(\theta, u) = (5\cosh u \cos\theta)\mathbf{i} + (5\cosh u \sin\theta)\mathbf{j} + (5\sinh u)\mathbf{k}.$$

Then $\mathbf{r}_\theta = (-5\cosh u \sin\theta)\mathbf{i} + (5\cosh u\cos\theta)\mathbf{j}$ and $\mathbf{r}_u = (5\sinh u\cos\theta)\mathbf{i} + (5\sinh u \sin\theta)\mathbf{j} + (5\cosh u)\mathbf{k}$, and thus

$$\begin{aligned}\mathbf{r}_\theta \times \mathbf{r}_u &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -5\cosh u\sin\theta & 5\cosh u\cos\theta & 0 \\ 5\sinh u\cos\theta & 5\sinh u \sin\theta & 5\cosh u \end{vmatrix} \\ &= (25\cosh^2 u\cos\theta)\mathbf{i} + (25\cosh^2 u \sin\theta)\mathbf{j} - 25(\cos u \sinh u)\mathbf{k}.\end{aligned}$$

At the point $(x_0, y_0, 0)$, where $x_0^2 + y_0^2 = 25$, we have $u = 0$ and $x_0 = 25\cos\theta$, $y_0 = 25\sin\theta$. So a normal for the tangent plane is $(25x_0)\mathbf{i} + (25y_0)\mathbf{j}$. Therefore the required tangent plane is

$$\begin{aligned} & (x_0\mathbf{i} + y_0\mathbf{j}) \cdot [(x - x_0)\mathbf{i} + (y - y_0\mathbf{j} + (z - 0)\mathbf{k}] = 0 \\ \Longrightarrow \quad & x_0 x + y_0 y = x_0^2 + y_0^2 = 25. \end{aligned}$$

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48. Find the area of the surface $2x^{3/2} + 2y^{3/2} - 3z = 0$ above the square $R : 0 \le x \le 1$, $0 \le y \le 1$, in the $xy$-plane.

**Solution.** The surface is given by the graph

$$z = f(x, y) := \frac{2}{3}x^{3/2} + \frac{2}{3}y^{3/2}, \qquad (x, y) \in R.$$

Then $\nabla f = \sqrt{x}\mathbf{i} + \sqrt{y}\mathbf{j}$ and $1 + |\nabla f|^2 = 1 + x + y$. Hence,

$$\begin{aligned}\text{Surface area} &= \iint_R \sqrt{1 + |\nabla f|^2}\, dA = \int_R \sqrt{1 + x + y}\, dx\, dy = \int_0^1 \int_0^1 \sqrt{1 + x + y}\, dx\, dy \\ &= \int_0^1 \left[\frac{2}{3}(1 + x + y)^{3/2}\right]_0^1 dy = \frac{2}{3} \cdot \frac{2}{5}\left[(2 + y)^{5/2} - (1 + y)^{5/2}\right]_0^1 \\ &= \frac{4}{15}\left(9\sqrt{3} - 8\sqrt{2} + 1\right).\end{aligned}$$

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## Exercises 16.6

4. Integrate the given function over the given surface.

   **Hemisphere** $G(x, y, z) = z^2$, over the hemisphere $x^2 + y^2 + z^2 = a^2$, $z \geq 0$.

   **Solution.** A parametrization is

$$\mathbf{r}(\phi, \theta) = (a \sin\phi \cos\theta)\mathbf{i} + (a \sin\phi \sin\theta)\mathbf{j} + (a\cos\phi)\mathbf{k}, \qquad 0 \leq \phi \leq \pi/2,\ 0 \leq \theta \leq 2\pi.$$

   Then $\mathbf{r}_\phi = (a\cos\phi\cos\theta)\mathbf{i} + (a\cos\phi\sin\theta)\mathbf{j} + (-a\sin\phi)\mathbf{k}$, $\mathbf{r}_\theta = (-a\sin\phi\sin\theta)\mathbf{i} + (a\sin\phi\cos\theta)\mathbf{j}$, and thus

$$\begin{aligned} \mathbf{r}_\phi \times \mathbf{r}_\theta &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a\cos\phi\cos\theta & a\cos\phi\sin\theta & -a\sin\phi \\ -a\sin\phi\sin\theta & a\sin\phi\cos\theta & 0 \end{vmatrix} \\ &= (a^2\sin^2\phi\cos\theta)\mathbf{i} + (a^2\sin^2\phi\sin\theta)\mathbf{j} + (a^2\cos\phi\sin\phi)\mathbf{k} \\ \implies \quad |\mathbf{r}_\phi \times \mathbf{r}_\theta| &= a^2 \sin\phi. \end{aligned}$$

   Hence,

$$\iint_S G(x, y, z)\, d\sigma = \int_0^{2\pi} \int_0^{\pi/2} (a^2\cos^2\phi)(a^2\sin\phi)\, d\phi\, d\theta = 2\pi a^4 \left[ -\frac{\cos^3\phi}{3} \right]_0^{\pi/2} = \frac{2}{3}\pi a^4.$$

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16. Integrate $G(x, y, z) = x$ over the surface given by

$$z = x^2 + y \quad \text{for} \quad 0 \leq x \leq 1, \quad -1 \leq y \leq 1.$$

   **Solution.** The surface is given by the graph

$$z = f(x, y) := x^2 + y, \qquad (x, y) \in \Omega := \{(x, y) : 0 \leq x \leq 1,\ -1 \leq y \leq 1\}.$$

   Then $\nabla f = 2x\mathbf{i} + \mathbf{j}$ and

$$\sqrt{1 + |\nabla f|^2} = \sqrt{4x^2 + 2}.$$

   Hence,

$$\begin{aligned} \iint_S G\, d\sigma &= \int_\Omega G(x, y, f(x, y)) \sqrt{1 + |\nabla f|^2}\, dx\, dy \\ &= \int_0^1 \int_{-1}^1 x \cdot \sqrt{4x^2 + 2}\, dy\, dx \\ &= 2 \cdot \frac{1}{8} \cdot \frac{2}{3} \left[ (4x^2 + 2)^{3/2} \right]_0^1 \\ &= \frac{3\sqrt{6} - \sqrt{2}}{3}. \end{aligned}$$

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24. Use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ across the surface in the specified direction.

   **Cylinder** $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ outward through the portion of the cylinder $x^2 + y^2 = 1$ cut by the planes $z = 0$ and $z = a$.

**Solution.** A parametrization is

$$\mathbf{r}(\theta, z) = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}, \qquad (x, y) \in \Omega := \{(x, y) : 0 \le \theta \le 2\pi,\ 0 \le z \le a\}.$$

Then $\mathbf{r}_\theta = (-\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$ and $\mathbf{r}_z = \mathbf{k}$, and thus

$$\mathbf{r}_\theta \times \mathbf{r}_z = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} \quad \text{(outward normal)},$$

$$\mathbf{F}(\mathbf{r}(\theta, z)) \cdot (\mathbf{r}_\theta \times \mathbf{r}_z) = ((\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + z\mathbf{k}) \cdot ((\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j}) = 1.$$

Hence,

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_\Omega \mathbf{F}(\mathbf{r}(\theta, z)) \cdot \frac{\mathbf{r}_\theta \times \mathbf{r}_z}{|\mathbf{r}_\theta \times \mathbf{r}_z|} |\mathbf{r}_\theta \times \mathbf{r}_z|\, dz\, d\theta \\ &= \int_0^{2\pi} \int_0^a 1\, dz\, d\theta = 2\pi a. \end{aligned}$$

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28. Use a parametrization to find the flux $\iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma$ across the surface in the specified direction.

**Paraboloid** $\mathbf{F} = 4x\mathbf{i} + 4y\mathbf{j} + 2\mathbf{k}$ outward (normal away from the $z$-axis) through the surface cut from the bottom of the paraboloid $z = x^2 + y^2$ by the plane $z = 1$.

**Solution.** A parametrization is

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}, \qquad 0 \le r \le 1,\ 0 \le \theta \le 2\pi.$$

Then $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$, and thus

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\mathbf{i} + (-2r^2\sin\theta)\mathbf{j} + r\mathbf{k}.$$

An outward normal is $-(\mathbf{r}_r \times \mathbf{r}_\theta) = (2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} - r\mathbf{k}$, and

$$\begin{aligned} \mathbf{F}(\mathbf{r}(r, \theta)) \cdot (-\mathbf{r}_r \times \mathbf{r}_\theta) &= ((4r\cos\theta)\mathbf{i} + (4r\sin\theta)\mathbf{j} + 2\mathbf{k}) \cdot \big((2r^2\cos\theta)\mathbf{i} + (2r^2\sin\theta)\mathbf{j} - r\mathbf{k}\big) \\ &= 8r^3 - 2r. \end{aligned}$$

Hence,

$$\begin{aligned} \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_\Omega \mathbf{F}(\mathbf{r}(r, \theta)) \cdot \frac{-(\mathbf{r}_r \times \mathbf{r}_\theta)}{|\mathbf{r}_r \times \mathbf{r}_\theta|} |\mathbf{r}_r \times \mathbf{r}_\theta|\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (8r^3 - 2r)\, dr\, d\theta = 2\pi. \end{aligned}$$

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