MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 7

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.3

25. **Independence of path** Show that the values of the integral $\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$ does not depend on the path taken from A to B.

Solution. Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = z^2\mathbf{i} + 2y\mathbf{j} + 2xz\mathbf{k}$. Since F is C^1 on $\mathbb{R}^3$ and satisfies

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 2z = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 0 = \frac{\partial M}{\partial y},$$

$\mathbf{F}$ is conservative. Therefore, $\int_A^B z^2\,dx + 2y\,dy + 2xz\,dz$ is independent of the path taken from A to B. ◀

31. **Evaluating a work integral tow ways** Let $\mathbf{F} = \nabla(x^3y^2)$ and let C be the path in the xy-plane from $(-1,1)$ to $(1,1)$ that consists of the line segment from $(-1,1)$ to $(0,0)$ followed by the line segment from $(0,0)$ to $(1,1)$. Evaluate $\int_C \mathbf{F}\cdot d\mathbf{r}$ in two ways.

   (a) Find parametrizations for the segment that make up C and evaluate the integral.

   (b) Use $f(x,y) = x^3y^2$ as a potential function for $\mathbf{F}$.

Solution. (a) $\mathbf{F} = \nabla(x^3y^2) = 3x^2y^2\mathbf{i} + 2x^3y\mathbf{j}$.

Line segment C_1 from $(-1,1)$ to $(0,0)$: $\mathbf{r}_1(t) = (t-1)\mathbf{i} + (1-t)\mathbf{j}, \quad 0 \le 1 \le t$. Then $\mathbf{r}_1'(t) = \mathbf{i} - \mathbf{j}$, and $\mathbf{F}\cdot\mathbf{r}_1' = 3(t-1)^2(1-t)^2 - 2(t-1)^3(1-t) = 5(t-1)^4$.

Line segment C_2 from $(0,0)$ to $(1,1)$ to : $\mathbf{r}_2(t) = t\mathbf{i} + t\mathbf{j}, \quad 0 \le 1 \le t$. Then $\mathbf{r}_2'(t) = \mathbf{i} + \mathbf{j}$, and $\mathbf{F}\cdot\mathbf{r}_2' = 3t^2t^2 + 2t^3t = 5t^4$.

Hence,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_{C_1} \mathbf{F}\cdot d\mathbf{r}_1 + \int_{C_2} \mathbf{F}\cdot d\mathbf{r}_2 = \int_0^1 5(t-1)^4\,dt + \int_0^1 5t^4\,dt = 2.$$

(b) Since $f(x,y) = x^3y^2$ is a potential function for $\mathbf{F}$,

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(1,1) - f(-1,1) = 2.$$

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Exercises 16.4

4. Verify the conclusion of Green's Theorem by evaluating both sides of Equations (3) and (4) for the field $\mathbf{F} = M\mathbf{i} + N\mathbf{j} = -x^2y\mathbf{i} + xy^2\mathbf{j}$. Take the domains of integration to be the disk R: $x^2 + y^2 \leq a^2$ and its bounding circle C: $\mathbf{r} = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \leq t \leq 2\pi$.

   Solution. Along the circle C,

$$M = -a^3\cos^2 t\sin t, \quad N = a^3\cos t\sin^2 t, \quad dx = -a\sin t\,dt, \quad dy = a\cos t\,dt.$$

   In the region R,

$$\frac{\partial M}{\partial x} = -2xy, \quad \frac{\partial M}{\partial y} = -x^2, \quad \frac{\partial N}{\partial x} = y^2, \quad \frac{\partial N}{\partial y} = 2xy.$$

   Equation (3):

$$\oint_C M\,dy - N\,dx = \int_0^{2\pi}(-a^4\cos^3 t\sin t + a^4\cos t\sin^3 t)\,dt = a^4\left[\frac{\cos^4 t}{4} + \frac{\sin^4 t}{4}\right]_0^{2\pi} = 0,$$

$$\iint_R\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)dx\,dy = \iint_R(-2xy + 2xy)\,dx\,dy = 0.$$

   Equation (4):

$$\begin{aligned}\oint_C M\,dx + N\,dy &= \int_0^{2\pi}(a^4\cos^2 t\sin^2 t + a^4\cos^2 t\sin^2 t)\,dt = \frac{a^4}{2}\int_0^{2\pi}\sin^2 2t\,dt\\ &= \frac{a^4}{4}\int_0^{4\pi}\sin^2 u\,du = \frac{\pi a^4}{2},\end{aligned}$$

$$\iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy = \iint_R(y^2 + x^2)\,dx\,dy = \int_0^{2\pi}\int_0^a r^2\cdot r\,dr\,d\theta = \frac{\pi a^4}{2}.$$

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7. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

   $\mathbf{F} = (y^2 - x^2)\mathbf{i} + (x^2 + y^2)\mathbf{j}$
   C: The triangle bounded by $y = 0$, $x = 3$, and $y = x$

   Solution. For $M := y^2 - x^2$, $N := x^2 + y^2$, we have

$$\frac{\partial M}{\partial x} = -2x, \quad \frac{\partial M}{\partial y} = 2y, \quad \frac{\partial N}{\partial x} = 2x, \quad \frac{\partial N}{\partial y} = 2y.$$

   Let R be the region enclosed by the curve C in the plane. By Green's Theorem,

$$\begin{aligned}\text{Couterclockwise circulation} &= \oint_C M\,dx + N\,dy = \iint_R\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)dx\,dy\\ &= \iint_R(2x - 2y)\,dx\,dy = \int_0^3\int_0^x(2x - 2y)\,dy\,dx = \int_0^3 x^2\,dx = 9;\end{aligned}$$

$$\text{Outward flux} = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$
$$= \iint_R (-2x + 2y)\,dx\,dy = -9.$$

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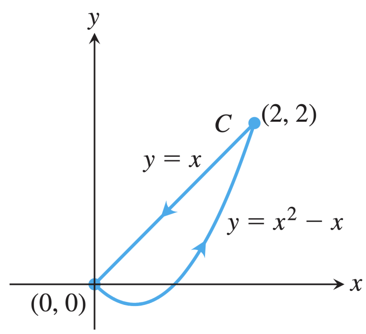
11. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field **F** and curve C.

$\mathbf{F} = x^3y^2\mathbf{i} + \frac{1}{2}x^4y\mathbf{j}$



Solution. For $M := x^3y^2$, $N := \frac{1}{2}x^4y$, we have

$$\frac{\partial M}{\partial x} = 3x^2y^2 \quad \frac{\partial M}{\partial y} = 2x^3y, \quad \frac{\partial N}{\partial x} = 2x^3y, \quad \frac{\partial N}{\partial y} = \frac{1}{2}x^4.$$

Let R be the region enclosed by the curve C in the plane. By Green's Theorem,

$$\text{Couterclockwise circulation} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$
$$= \iint_R (2x^3y - 2x^3y)\,dx\,dy = 0;$$

$$\text{Outward flux} = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$
$$= \iint_R (3x^2y^2 + \frac{1}{2}x^4)\,dx\,dy = \int_0^2 \int_{x^2-x}^{x} (3x^2y^2 + \frac{1}{2}x^4)\,dy\,dx$$
$$= \int_0^2 \left(x^5 - x^2(x^2-x)^3 + \frac{1}{2}x^5 - \frac{1}{2}x^4(x^2-x)\right) dx$$
$$= \int_0^2 \left(3x^5 - \frac{7}{2}x^6 + 3x^7 - x^8\right) dx = \frac{64}{9}.$$

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22. Apply Green's Theorem to evaluate the integral.

$\oint_C (3y\,dx + 2x\,dy) \qquad C$: The boundary of $0 \le x \le \pi$, $0 \le y \le \sin x$.

Solution. For $M := 3y$, $N := 2x$, we have

$$\frac{\partial M}{\partial y} = 3, \quad \frac{\partial N}{\partial x} = 2.$$

Let R be the region enclosed by the curve C in the plane. By Green's Theorem,

$$\oint_C (3y\,dx + 2x\,dy) = \iint_R (2-3)\,dx\,dy = -\int_0^\pi \int_0^{\sin x} dy\,dx = -\int_0^\pi \sin x\,dx = -2.$$

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24. Apply Green's Theorem to evaluate the integral.

$\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy$

C: Any simple closed curve in the plane for which Green's Theorem holds.

Solution. Let R be the region enclosed by the curve C in the plane. By Green's Theorem,

$$\begin{aligned}\oint_C (2x + y^2)\,dx + (2xy + 3y)\,dy &= \iint_R \left[\frac{\partial}{\partial x}(2xy + 3y) - \frac{\partial}{\partial y}\left(2x + y^2\right)\right] dx\,dy \\ &= \iint_R (2y - 2y)\,dx\,dy = 0.\end{aligned}$$

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25. Use the Green's Theorem area formula to find the area of the region enclosed by the curve.

The circle $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$

Solution. By Green's Theorem area formula,

$$\begin{aligned}\text{Area} = \frac{1}{2}\oint_C x\,dy - y\,dx &= \frac{1}{2}\int_0^{2\pi} \left[(a\cos t)(a\cos t) - (a\sin t)(-a\sin t)\right] dt \\ &= \frac{1}{2}\int_0^{2\pi} a^2\,dt = \pi a^2.\end{aligned}$$

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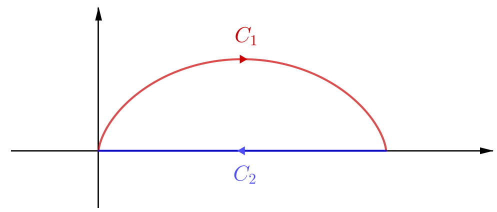
28. Use the Green's Theorem area formula to find the area of the region enclosed by the curve.

One arch of the cycloid $x = t - \sin t, \quad y = 1 - \cos t.$

Solution. Let $C_1 : (x, y) = (t - \sin t, 1 - \cos t), \quad 0 \le t \le 2\pi$ and $C_2 : (x, y) = (2\pi - t, 0), \quad 0 \le t \le 2\pi.$



Since $C \coloneqq C_1 \cup C_2$ is traversed clockwise, the area enclosed by C is given by

$$\begin{aligned}
\text{Area} &= \frac{1}{2}\oint_{-C} x\,dy - y\,dx = -\frac{1}{2}\oint_{C} x\,dy - y\,dx \\
&= -\frac{1}{2}\oint_{C_1} x\,dy - y\,dx - \frac{1}{2}\oint_{C_2} x\,dy - y\,dx \\
&= -\frac{1}{2}\int_0^{2\pi} (0)\,dt - \frac{1}{2}\int_0^{2\pi} \left[(t-\sin t)(\sin t) - (1-\cos t)(1-\cos t)\right]\,dt \\
&= 0 - \frac{1}{2}\int_0^{2\pi} \left(t\sin t - 2 + 2\cos t\right)\,dt \\
&= -\frac{1}{2}\left(-2\pi - 4\pi + 0\right) = 3\pi.
\end{aligned}$$

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30. **Integral dependent only on area** Show that the value of

$$\oint_C xy^2\,dx + (xy^2 + 2x)\,dy$$

around any square depends only on the area of the square and not on its location in the plane.

Solution. Let R be the region enclosed by the curve C in the plane. By Green's Theorem,

$$\begin{aligned}
\oint_C xy^2\,dx + (x^2y + 2x)\,dy &= \iint_R \left[\frac{\partial}{\partial x}\left(x^2y + 2x\right) - \frac{\partial}{\partial y}\left(xy^2\right)\right]\,dx\,dy \\
&= \iint_R (2xy + 2 - 2xy)\,dx\,dy \\
&= 2\iint_R \,dx\,dy \\
&= 2 \times \text{area of the square.}
\end{aligned}$$

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