MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 6

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.2

41. **A field of tangent vectors**

   (a) Find a field $\mathbf{G} = P(x, y)\mathbf{i} + Q(x, y)\mathbf{j}$ in the xy-plane with the property that at any point $(a, b) \neq (0, 0)$, G is a vector of magnitude $\sqrt{a^2 + b^2}$ tangent to the circle $x^2+y^2 = a^2+b^2$ and pointing in the couterclockwise direction. (The field is undefined at $(0, 0)$.)

   (b) How is $\mathbf{G}$ related to the spin vector field $\mathbf{F}$ in Figure 16.12?

   Solution. (a) $x^2 + y^2 = a^2 + b^2 \implies 2x + 2yy' = 0 \implies y' = -\frac{x}{y}$ is the slope of the tangent line at any point on the circle $\implies y' = -\frac{a}{b}$ at (a, b). Let $\mathbf{v} = -b\mathbf{i} + a\mathbf{j} \implies |\mathbf{v}| = \sqrt{a^2 + b^2}$, with $\mathbf{v}$ in a counterclockwise direction and tangent to the circle. So we have $\mathbf{G} = -y\mathbf{i} + x\mathbf{j}$.

   (b) $\mathbf{G} = (\sqrt{x^2 + y^2})\mathbf{F} = (\sqrt{a^2 + b^2})\mathbf{F}$.

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44. **Two "central" fields** Find a field $\mathbf{F} = M(x, y)\mathbf{i} + N(x, y)\mathbf{j}$ in the xy-plane with the property that at each point $(x, y) \neq (0, 0)$, $\mathbf{F}$ points toward the origin and $|\mathbf{F}|$ is

   (a) the distance from (x, y) to the origin,

   (b) inversely proportional to the distance from (x, y) to the origin. (The field is undefined at $(0, 0)$.)

   Solution. (a) $-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}$ is a unit vector through (x, y) pointing toward the origin. Since $|\mathbf{F}| = \sqrt{x^2 + y^2}$, we have $\mathbf{F} = \sqrt{x^2 + y^2}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -x\mathbf{i} - y\mathbf{j}$.

   (b) We want $|\mathbf{F}| = \frac{C}{\sqrt{x^2+y^2}}$, where $C > 0$ is a constant $\implies \mathbf{F} = \frac{C}{\sqrt{x^2+y^2}}\left(-\frac{x\mathbf{i}+y\mathbf{j}}{\sqrt{x^2+y^2}}\right) = -C\left(\frac{x\mathbf{i}+y\mathbf{j}}{x^2+y^2}\right)$.

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49. $\mathbf{F}$ is the velocity field of a fluid flowing through a region in space. Find the flow along the given curve in the direction of increasing t.

   $\mathbf{F} = (x - z)\mathbf{i} + x\mathbf{k}$; $\quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{k}$, $\quad 0 \leq t \leq \pi$.

   Solution. $\mathbf{F} = (\cos t - \sin t)\mathbf{i} + (\cos t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{k} \implies \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\sin t\cos t + 1$. Hence,

$$\text{Flow} = \int_0^\pi (-\sin t\cos t + 1)\, dt = \pi.$$

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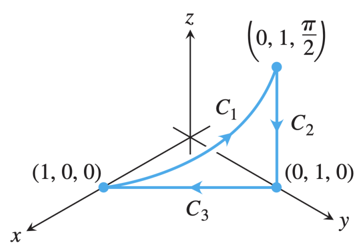
51. **Circulation** Find the circulation of $\mathbf{F} = 2x\mathbf{i} + 2z\mathbf{j} + 2y\mathbf{k}$ around the closed path consisting of the following three curves traversed in the direction of increasing t.

$$\begin{aligned} C_1 &: \quad \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi/2 \\ C_2 &: \quad \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1-t)\mathbf{k}, \quad 0 \le t \le 1 \\ C_3 &: \quad \mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}, \quad 0 \le t \le 1. \end{aligned}$$



Solution. $C_1 : \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le \pi/2 \implies \mathbf{F} = (2\cos t)\mathbf{i} + 2t\mathbf{j} + (2\sin t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j} + \mathbf{k} \implies \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -2\cos t \sin t + 2t\cos t + 2\sin t$. Hence,

$$\text{Flow}_1 = \int_0^{\pi/2} (-2\cos t \sin t + 2t\cos t + 2\sin t)\, dt = \left[\frac{1}{2}\cos 2t + 2t\sin t\right]_0^{\pi/2} = -1 + \pi.$$

$C_2 : \mathbf{r}(t) = \mathbf{j} + (\pi/2)(1-t)\mathbf{k}, \quad 0 \le t \le 1 \implies \mathbf{F} = \pi(1-t)\mathbf{j} + 2\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = -\pi/2\mathbf{k}$ $\implies \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = -\pi$. Hence,

$$\text{Flow}_2 = \int_0^1 (-\pi)\, dt = -\pi.$$

$C_3 : \mathbf{r}(t) = t\mathbf{i} + (1-t)\mathbf{j}, \quad 0 \le t \le 1 \implies \mathbf{F} = 2t\mathbf{i} + 2(1-t)\mathbf{k}$ and $\frac{d\mathbf{r}}{dt} = \mathbf{i} - \mathbf{j} \implies \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = 2t$. Hence,

$$\text{Flow}_3 = \int_0^1 (2t)\, dt = 1.$$

Therefore,

$$\text{Circulation} = \text{Flow}_1 + \text{Flow}_2 + \text{Flow}_3 = (-1+\pi) + (-\pi) + 1 = 0.$$

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54. **Flow of a gradient field** Find the flow of the field $\mathbf{F} = \nabla(xy^2z^3)$:

(a) Once around the curve C in Exercise 52, clockwise as viewed from above.

(b) Along the line segment from $(1,1,1)$ to $(2,1,-1)$.

Solution. (a) $\mathbf{F} = \nabla(xy^2z^3) \implies \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \frac{d}{dt} f(\mathbf{r}(t))$, where $f(x,y,z) = xy^2z^3$. Hence,

$$\text{Flow} = \oint_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \int_a^b \frac{d}{dt} f(\mathbf{r}(t)) = f(\mathbf{r}(b)) - f(\mathbf{r}(a)) = 0, \quad \text{since } C \text{ is an entire ellipse.}$$

(b)

$$\text{Flow} = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dt} = \int_{(1,1,1)}^{(2,1,-1)} \frac{d}{dt} f(\mathbf{r}(t))\, dt = \left[xy^2z^3\right]_{(1,1,1)}^{(2,1,-1)} = -2 - 1 = -3.$$

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Exercises 16.3

5. Is the field $\mathbf{F} = (z+y)\mathbf{i} + z\mathbf{j} + (y+x)\mathbf{k}$ conservative or not?

Solution. Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = (z+y)\mathbf{i} + z\mathbf{j} + (y+x)\mathbf{k}$.
Since $\frac{\partial M}{\partial y} = 1 \neq 0 = \frac{\partial N}{\partial x}$, $\mathbf{F}$ is not conservative. ◀

6. Is the field $\mathbf{F} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j} + z\mathbf{k}$ conservative or not?

Solution. Write $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k} = (e^x \cos y)\mathbf{i} - (e^x \sin y)\mathbf{j} + z\mathbf{k}$. Clearly, F is C^1 on $\mathbb{R}^3$, which is connected and simply connected. Since

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = -e^x \sin y = \frac{\partial M}{\partial y},$$

$\mathbf{F}$ is conservative. ◀

10. Find a potential function f for the field $\mathbf{F} = (y \sin z)\mathbf{i} + (x \sin z)\mathbf{j} + (xy \cos z)\mathbf{k}$.

Solution. Suppose f is a potential function, that is $\nabla f = \mathbf{F}$. Then

$$\frac{\partial f}{\partial x} = y \sin z \implies f(x,y,z) = xy \sin z + g(y,z);$$

$$\frac{\partial f}{\partial y} = x \sin z + \frac{\partial g}{\partial y} = x \sin z \implies \frac{\partial g}{\partial y} = 0 \implies g(y,z) = h(z);$$

$$\frac{\partial f}{\partial z} = xy \cos z + h'(z) = xy \cos z \implies h'(z) = 0 \implies h(z) = C.$$

Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = xy \sin z + C$, where C is a constant. ◀

15. Show that the differential form in the integral is exact. Then evaluate the integral.

$$\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz.$$

Solution. Denote the differential form as $M\,dx + N\,dy + P\,dz$. Clearly, it is C^1 on $\mathbb{R}^3$, which is connected and simply connected. It is exact because

$$\frac{\partial P}{\partial y} = -2z = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 2x = \frac{\partial M}{\partial y}.$$

Suppose $\nabla f = 2xy\mathbf{i} + (x^2 - z^2)\sin x\mathbf{j} - 2yz\mathbf{k}$. Then

$$\frac{\partial f}{\partial x} = 2xy \implies f(x,y,z) = x^2 y + g(y,z);$$

$$\frac{\partial f}{\partial y} = x^2 + \frac{\partial g}{\partial y} = x^2 - z^2 \implies \frac{\partial g}{\partial y} = -z^2 \implies g(y,z) = -yz^2 + h(z);$$

$$\frac{\partial f}{\partial z} = -2yz + h'(z) = -2yz \implies h'(z) = 0 \implies h(z) = C.$$

Hence, $f(x,y,z) = x^2 y - yz^2 + C$, where C is a constant. Therefore,

$$\int_{(0,0,0)}^{(1,2,3)} 2xy\,dx + (x^2 - z^2)\,dy - 2yz\,dz = f(1,2,3) - f(0,0,0) = -16.$$

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18. Find a potential function for the field and evaluate the integral as in Example 6.

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)\,dy + \frac{1}{z}\,dz.$$

Solution. Denote the differential form as $M\,dx + N\,dy + P\,dz$. Note that $D := \{(x, y, z) \in \mathbb{R}^3 : y > 0, z > 0\}$ is open, connected and simply connected, and it contains the points $(0, 2, 1)$, $(1, \pi/2, 2)$. Clearly the differential form is C^1 on D. It is exact because

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = -2\sin y = \frac{\partial M}{\partial y}.$$

Suppose $\nabla f = 2\cos y\mathbf{i} + \left(\frac{1}{y} - 2x\sin y\right)\mathbf{j} + \frac{1}{z}\mathbf{k}$. Then

$$\frac{\partial f}{\partial x} = 2\cos y \implies f(x, y, z) = 2x\cos y + g(y, z);$$

$$\frac{\partial f}{\partial y} = -2x\sin y + \frac{\partial g}{\partial y} = \frac{1}{y} - 2x\sin y \implies \frac{\partial g}{\partial y} = \frac{1}{y} \implies g(y, z) = \ln|y| + h(z);$$

$$\frac{\partial f}{\partial z} = h'(z) = \frac{1}{z} \implies h(z) = \ln|z|.$$

Hence, $f(x, y, z) = 2x\cos y + \ln|y| + \ln|z| + C$, where C is a constant. Therefore,

$$\int_{(0,2,1)}^{(1,\pi/2,2)} 2\cos y\,dx + \left(\frac{1}{y} - 2x\sin y\right)\,dy + \frac{1}{z}\,dz = f(1, \pi/2, 2) - f(0, 2, 1) = \ln\frac{\pi}{2}.$$

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