MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 5

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercise 16.1

12. Evaluate $\int_C \sqrt{x^2+y^2}\,ds$ along the curve $\mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k}$, $-2\pi \le t \le 2\pi$.

Solution. $C : \mathbf{r}(t) = (4\cos t)\mathbf{i} + (4\sin t)\mathbf{j} + 3t\mathbf{k},\ -2\pi \le t \le 2\pi \implies \frac{d\mathbf{r}}{dt} = (-4\sin t)\mathbf{i} + (4\cos t)\mathbf{j} + 3\mathbf{k} \implies \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{16\sin^2 t + 16\cos^2 t + 9} = 5$; $\sqrt{x^2+y^2} = \sqrt{16\sin^2 t + 16\cos^2 t} = 4$. Hence,

$$\int_C \sqrt{x^2+y^2}\,ds = \int_{-2\pi}^{2\pi} (4)(5)\,dt = [20t]_{-2\pi}^{2\pi} = 80\pi.$$

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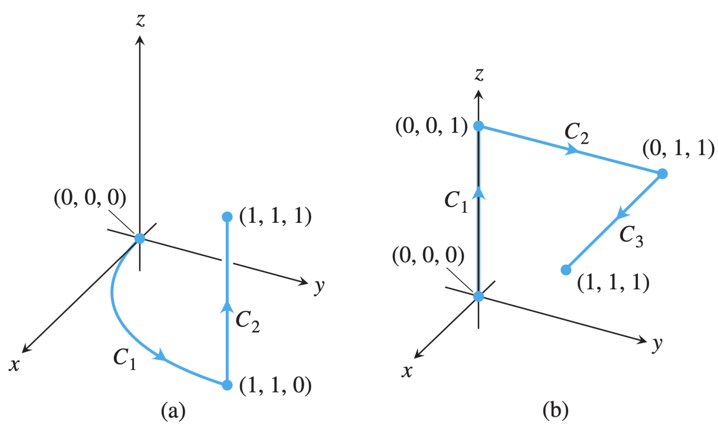
16. Integrate $f(x,y,z) = x + \sqrt{y} - z^2$ over the path from $(0,0,0)$ to $(1,1,1)$ (see the accompanying figure) given by

$$C_1 : \mathbf{r}(t) = t\mathbf{k}, \quad 0 \le t \le 1$$
$$C_2 : \mathbf{r}(t) = t\mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1$$
$$C_3 : \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad 0 \le t \le 1.$$



The paths of integration for Exercises 15 and 16.

Solution. $C_1 : \mathbf{r}(t) = t\mathbf{k}\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{k} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$; and $f(\mathbf{r}(t)) = -t^2$. So

$$\int_{C_1} f(x,y,z)\,ds = \int_0^1 (-t^2)(1)\,dt = -\frac{1}{3}.$$

$C_2 : \mathbf{r}(t) = t\mathbf{j} + \mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$; and $f(\mathbf{r}(t)) = \sqrt{t} - 1$. So

$$\int_{C_2} f(x,y,z)\,ds = \int_0^1 (\sqrt{t}-1)(1)\,dt = -\frac{1}{3}.$$

$C_3 : \mathbf{r}(t) = \mathbf{r}(t) = t\mathbf{i} + \mathbf{j} + \mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$; and $f(\mathbf{r}(t)) = t$. So

$$\int_{C_3} f(x,y,z)\,ds = \int_0^1 (t)(1)\,dt = \frac{1}{2}.$$

Therefore, $\int_C f(x,y,z)\,ds = \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \int_{C_3} f(x,y,z)\,ds = -\frac{1}{3} - \frac{1}{3} + \frac{1}{2} = -\frac{1}{6}$. ◀

17. Integrate $f(x,y,z) = (x+y+z)/(x^2+y^2+z^2)$ over the path $\mathbf{r}(t) = t\mathbf{i}+t\mathbf{j}+t\mathbf{k}$, $0 < a \le t \le b$.

Solution. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 < a \le t \le b \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k} \implies \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{3}$; $f(\mathbf{r}(t)) = \frac{t+t+t}{t^2+t^2+t^2} = \frac{1}{t}$. Hence,

$$\int_C f(x,y,z)\,ds = \int_a^b (\frac{1}{t})\sqrt{3}\,dt = \sqrt{3}\ln(\frac{b}{a}).$$

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24. Find the line integral of $f(x,y) = \sqrt{y}/x$ along the curve $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j}$, $1/2 \le t \le 1$.

Solution. $\mathbf{r}(t) = t^3\mathbf{i} + t^4\mathbf{j},\ 1/2 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = 3t^2\mathbf{i} + 4t^3\mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = t^2\sqrt{9+16t^2}$; $f(\mathbf{r}(t)) = \frac{\sqrt{t^4}}{t^3} = \frac{1}{t}$. Hence,

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_{1/2}^1 (\frac{1}{t})t^2\sqrt{9+16t^2}\,dt = \int_{1/2}^1 t\sqrt{9+16t^2}\,dt \\ &= \left[\frac{1}{48}(9+16t^2)^{3/2}\right]_{1/2}^1 = \frac{125 - 13\sqrt{13}}{48}.\end{aligned}$$

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32. Find the area of one side of the "wall" standing orthogonally on the curve $2x + 3y = 6$, $0 \le x \le 6$, and beneath the curve on the surface $f(x,y) = 4 + 3x + 2y$.

Solution. $2x + 3y = 6,\ 0 \le x \le 6 \implies \mathbf{r}(t) = t\mathbf{i} + (2 - \frac{2}{3}t)\mathbf{j},\ 0 \le t \le 6 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} - \frac{2}{3}\mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = \frac{\sqrt{13}}{3}$. Hence,

$$\text{Area} = \int_C f(x,y)\,ds = \int_0^6 (4 + 3t + 2(2 - \frac{2}{3}t))\frac{\sqrt{13}}{3}\,dt = \frac{\sqrt{13}}{3}\int_0^6 (8 + \frac{5}{3}t)\,dt = 26\sqrt{13}.$$

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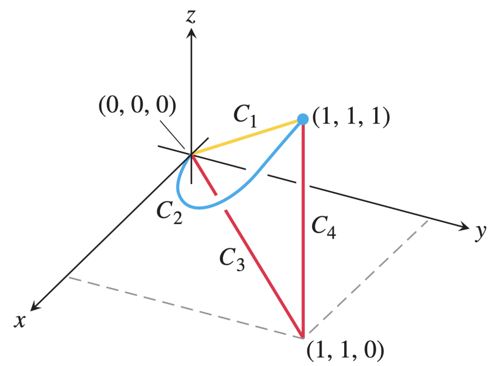
Exercise 16.2

10. Find the line integral of $\mathbf{F} = xy\mathbf{i} + yz\mathbf{j} + xz\mathbf{k}$ from $(0,0,0)$ to $(1,1,1)$ over each of the following paths in the accompanying figure.

(a) The straight-line path $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}$, $\quad 0 \le t \le 1$.

(b) The curved path $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}$, $\quad 0 \le t \le 1$.

(c) The path $C_3 \cup C_4$ consisting of the line segment from $(0,0,0)$ to $(1,1,0)$ followed by the segment from $(1,1,0)$ to $(1,1,1)$.



Solution. (a) $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence,

$$\int_{C_1} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 (t^2\mathbf{i} + t^2\mathbf{j} + t^2\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt$$
$$= \int_0^1 3t^2\, dt = \left[t^3\right]_0^1 = 1.$$

(b) $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}$. Hence,

$$\int_{C_2} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 (t^3\mathbf{i} + t^6\mathbf{j} + t^5\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k})\, dt$$
$$= \int_0^1 (t^3 + 2t^7 + 4t^8)\, dt = \left[\frac{t^4}{4} + \frac{t^8}{4} + \frac{4t^9}{9}\right]_0^1 = \frac{17}{18}.$$

(c) $C_3 : \mathbf{r}_3(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}_3}{dt} = \mathbf{i} + \mathbf{j} \implies \mathbf{F}(\mathbf{r}_3(t)) \cdot \frac{d\mathbf{r}_3}{dt} = (t^2\mathbf{i}) \cdot (\mathbf{i} + \mathbf{j}) = t^2$;
$C_4 : \mathbf{r}_4(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}_4}{dt} = \mathbf{k} \implies \mathbf{F}(\mathbf{r}_4(t)) \cdot \frac{d\mathbf{r}_4}{dt}(\mathbf{i} + t\mathbf{j} + t\mathbf{k}) \cdot \mathbf{k} = t$.
Hence,

$$\int_{C_3 \cup C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 t^2\, dt + \int_0^1 t\, dt = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

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22. Find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.
$\mathbf{F} = 6z\mathbf{i} + y^2\mathbf{j} + 12x\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi.$

Solution. $C : \mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + (t/6)\mathbf{k},\ 0 \le t \le 2\pi \implies \frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} + (-\sin t)\mathbf{j} + \frac{1}{6}\mathbf{k}$; $\mathbf{F}(\mathbf{r}(t)) = t\mathbf{i} + (\cos^2 t)\mathbf{j} + (12 \sin t)\mathbf{k}$; and so $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = t\cos t - \sin t \cos^2 t + 2\sin t$.
Hence,

$$\text{Work done by } \mathbf{F} = \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (t\cos t - \sin t\cos^2 t + 2\sin t)\, dt$$
$$= \left[\cos t + t\sin t + \frac{1}{3}\cos^3 t - 2\cos t\right]_0^{2\pi} = 0.$$

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24. Evaluate $\int_C (x-y)\,dx+(x+y)\,dy$ counterclockwise around the triangle with vertices $(0,0)$, $(1,0)$, and $(0,1)$.

Solution. Let $\mathbf{F} = (x-y)\mathbf{i} + (x+y)\mathbf{j}$.

Along $(0,0)$ to $(1,0)$: $\mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 1$, and $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = (t\mathbf{i} + t\mathbf{j}) \cdot \mathbf{i} = t$;

Along $(1,0)$ to $(0,1)$: $\mathbf{r}(t) = (1-t)\mathbf{i}+t\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F}(\mathbf{r}(t))\cdot \frac{d\mathbf{r}}{dt} = ((1-2t)\mathbf{i}+\mathbf{j})\cdot(-\mathbf{i}+\mathbf{j}) = 2t$;

Along $(0,1)$ to $(0,0)$: $\mathbf{r}(t) = (1-t)\mathbf{j},\ 0 \le t \le 1$, and $\mathbf{F}(\mathbf{r}(t))\cdot \frac{d\mathbf{r}}{dt} = ((t-1)\mathbf{i}+(1-t)\mathbf{j})\cdot(-\mathbf{j}) = t-1$.

Hence,

$$\int_C (x-y)\,dx + (x+y)\,dy = \int_0^1 t\,dt + \int_0^1 2t\,dt + \int_0^1 (t-1)\,dt$$
$$= \int_0^1 (4t-1)\,dt = 1.$$

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35. **Flow integrals** Find the flow of the velocity field $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$ along each of the following paths from $(1,0)$ to $(-1,0)$ in the xy-plane.

(a) The upper half of the circle $x^2 + y^2 = 1$.
(b) The line segment from $(1,0)$ to $(-1,0)$.
(c) The line segment from $(1,0)$ to $(0,-1)$ followed by the line segment from $(0,-1)$ to $(-1,0)$.

Solution. (a) $C_1 : \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le \pi \implies \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$;
$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = ((\cos t + \sin t)\mathbf{i} - \mathbf{j}) \cdot ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j}) = -\sin t\cos t - \sin^2 t - \cos t$.
Hence,

$$\text{Flow} = \int_{C_1} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^{\pi} (-\sin t\cos t - \sin^2 t - \cos t)\,dt = -\frac{\pi}{2}.$$

(b) $C_2 : \mathbf{r}(t) = (1-2t)\mathbf{i},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = -2\mathbf{i}$;
$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = \big((1-2t)\mathbf{i} - (1-2t)^2\mathbf{j}\big) \cdot (-2\mathbf{i}) = 4t - 2$.
Hence,

$$\text{Flow} = \int_{C_2} \mathbf{F}\cdot\mathbf{T}\,ds = \int_0^1 (4t-2)\,dt = 0.$$

(c) $C_3 : \mathbf{r}(t) = (1-t)\mathbf{i} - t\mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = -\mathbf{i} - \mathbf{j}$;
$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = \big((1-2t)\mathbf{i} - (1-2t+2t^2)\mathbf{j}\big) \cdot (-\mathbf{i} - \mathbf{j}) = 2t^2$.
$C_4 : \mathbf{r}(t) = -t\mathbf{i} + (t-1)\mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = -\mathbf{i} + \mathbf{j}$;
$\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = \big(-\mathbf{i} - (1-2t+2t^2)\mathbf{j}\big) \cdot (-\mathbf{i} + \mathbf{j}) = 2t - 2t^2$.
Hence,

$$\text{Flow} = \int_{C_3} \mathbf{F}\cdot\mathbf{T}\,ds + \int_{C_4} \mathbf{F}\cdot\mathbf{T}\,ds$$
$$= \int_0^1 2t^2\,dt + \int_0^1 (2t-2t^2)\,dt = \frac{2}{3} + \frac{1}{3} = 1.$$

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36. **Flux across a triangle** Find the flux of the field $\mathbf{F}$ in Exercise 35 outward across the triangle with vertices $(1, 0)$, $(0, 1)$, $(-1, 0)$.

Solution. From $(1, 0)$ to $(0, 1)$: $C_1 : \mathbf{r}_1(t) = (1 - t)\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1,\ \mathbf{r}_1'(t) = -\mathbf{i} + \mathbf{j}$; the outward unit normal $\mathbf{n}_1$ satisfies $\mathbf{n}_1|\mathbf{r}_1'(t)| = (-\mathbf{i} + \mathbf{j}) \times \mathbf{k} = \mathbf{i} + \mathbf{j}$. Hence,

$$\begin{aligned}\text{Flux}_1 &= \int_{C_1} \mathbf{F} \cdot \mathbf{n}_1\, ds = \int_0^1 \mathbf{F}(\mathbf{r}_1(t)) \cdot \mathbf{n}_1|\mathbf{r}_1'(t)|\, dt \\ &= \int_0^1 (\mathbf{i} - (1 - 2t + 2t^2)\mathbf{j}) \cdot (\mathbf{i} + \mathbf{j})\, dt \\ &= \int_0^1 (2t - 2t^2)\, dt = \frac{1}{3}.\end{aligned}$$

From $(0, 1)$ to $(-1, 0)$: $C_2 : \mathbf{r}_2(t) = -t\mathbf{i} + (1 - t)\mathbf{j},\ 0 \le t \le 1,\ \mathbf{r}_2'(t) = -\mathbf{i} - \mathbf{j}$; the outward unit normal $\mathbf{n}_2$ satisfies $\mathbf{n}_2|\mathbf{r}_2'(t)| = (-\mathbf{i} - \mathbf{j}) \times \mathbf{k} = -\mathbf{i} + \mathbf{j}$. Hence,

$$\begin{aligned}\text{Flux}_2 &= \int_{C_2} \mathbf{F} \cdot \mathbf{n}_2\, ds = \int_0^1 \mathbf{F}(\mathbf{r}_2(t)) \cdot \mathbf{n}_2|\mathbf{r}_2'(t)|\, dt \\ &= \int_0^1 ((1 - 2t)\mathbf{i} - (1 - 2t + 2t^2)\mathbf{j}) \cdot (-\mathbf{i} + \mathbf{j})\, dt \\ &= \int_0^1 (-2 + 4t - 2t^2)\, dt = -\frac{2}{3}.\end{aligned}$$

From $(-1, 0)$ to $(1, 0)$: $C_3 : \mathbf{r}_3(t) = (-1 + 2t)\mathbf{i},\ 0 \le t \le 1,\ \mathbf{r}_3'(t) = 2\mathbf{i}$; the outward unit normal $\mathbf{n}_3$ satisfies $\mathbf{n}_3|\mathbf{r}_3'(t)| = (2\mathbf{i}) \times \mathbf{k} = -2\mathbf{j}$. Hence,

$$\begin{aligned}\text{Flux}_3 &= \int_{C_3} \mathbf{F} \cdot \mathbf{n}_3\, ds = \int_0^1 \mathbf{F}(\mathbf{r}_3(t)) \cdot \mathbf{n}_3|\mathbf{r}_3'(t)|\, dt \\ &= \int_0^1 ((-1 + 2t)\mathbf{i} - (1 - 4t + 4t^2)\mathbf{j}) \cdot (-2\mathbf{j})\, dt \\ &= 2\int_0^1 (1 - 4t + 4t^2)\, dt = \frac{2}{3}.\end{aligned}$$

Therefore, $\text{Flux} = \text{Flux}_1 + \text{Flux}_2 + \text{Flux}_3 = \frac{1}{3} - \frac{2}{3} + \frac{2}{3} = \frac{1}{3}$. ◄