MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 4

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercise 15.7

35. Consider the solid enclosed by the cardioid of revolution $\rho = 1 - \cos\phi$.

    (a) Find the spherical coordinate limits for the integral that calculates the volume of the given solid.

    (b) Evaluate the integral.

Solution. Volume $= \int_0^{2\pi}\int_0^{\pi}\int_0^{1-\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_0^{\pi} \frac{1}{3}(1-\cos\phi)^3 \sin\phi \, d\phi \, d\theta$
$= \frac{1}{3}\int_0^{2\pi} \left[\frac{(1-\cos\phi)^4}{4}\right]_0^{\pi} d\theta = \frac{8\pi}{3}.$ ◀

37. Consider the solid bounded below by the sphere $\rho = 2\cos\phi$ and above by the cone $z = \sqrt{x^2+y^2}$.

    image

    (a) Find the spherical coordinate limits for the integral that calculates the volume of the given solid.

    (b) Evaluate the integral.

Solution. Volume $= \int_0^{2\pi}\int_{\pi/4}^{\pi/2}\int_0^{2\cos\phi} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = \int_0^{2\pi}\int_{\pi/4}^{\pi/2} \frac{8}{3}\cos^3\phi \sin\phi \, d\rho \, d\theta$
$= \frac{8}{3}\int_0^{2\pi}\left[-\frac{\cos^4\phi}{4}\right]_{\pi/4}^{\pi/2} d\theta = \frac{\pi}{3}.$ ◀

Exercise 15.8

2. (a) Solve the system

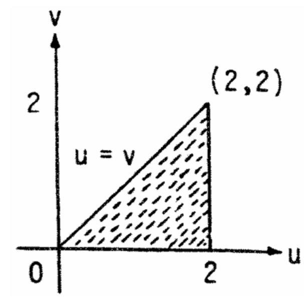
$$u = x + 2y, \qquad v = x - y$$

    for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x,y)/\partial(u,v)$.

    (b) Find the image under the transformation $u = x+2y$, $v = x - y$ of the triangular region in the xy-plane bounded by the line $y=0$, $y=x$, and $x+2y=2$. Sketch the transformed region in the uv-plane.

Solution. (a) $x = \frac{1}{3}(u+2v)$ and $y = \frac{1}{3}(u-v)$. $\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix} = -\frac{1}{9} - \frac{2}{9} = -\frac{1}{3}.$

(b) The triangular region in the xy-plane has vertices $(0,0)$, $(2,0)$ and $(\frac{2}{3},\frac{2}{3})$.
$y = 0 \implies u = v;\quad y = x \implies v = 0;\quad x+2y = 2 \implies u = 2.$



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4. (a) Solve the system

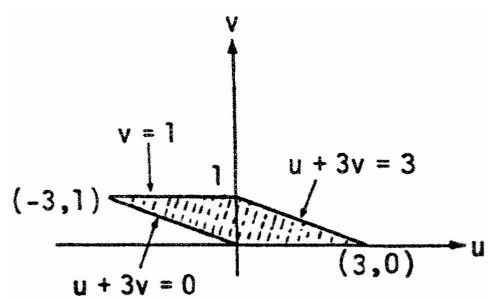
$$u = 2x - 3y, \qquad v = -x + y$$

for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

(b) Find the image under the transformation $u = 2x - 3y$, $v = -x + y$ of the parallelogram R the xy-plane with boundaries $x = -3$, $x = 0$, $y = x$, and $y = x + 1$. Sketch the transformed region in the uv-plane.

Solution. (a) $x = -u - 3v$ and $y = -u - 2v$. $\dfrac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} -1 & -3 \\ -1 & -2 \end{vmatrix} = 2 - 3 = -1.$

(b) $x = -3 \implies u + 3v = 3$; $x = 0 \implies u + 3v = 0$; $y = x \implies v = 0$; $y = x + 1 \implies v = 1$.



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8. Use the transformation and parallelogram R in Exercise 4 to evaluate the integral

$$\iint_R 2(x - y)\, dx\, dy.$$

Solution.

$$\begin{aligned} \iint_R 2(x - y)\, dx\, dy &= \iint_G -2v \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv \qquad \text{(the region } G \text{ is sketched in Exercise 4)} \\ &= \iint_G -2v\, du\, dv \\ &= \int_0^1 \int_{-3v}^{3-3v} -2v\, du\, dv \\ &= \int_0^1 -6v\, dv = -3. \end{aligned}$$

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9. Let R be the region in the first quadrant of the xy-plane bounded by the hyperbolas $xy=1$, $xy=9$ and the lines $y=x$, $y=4x$. Use the transformation $x=u/v$, $y=uv$ with $u>0$ and $v>0$ to rewrite

$$\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)\,dx\,dy.$$

Solution. $x=\frac{u}{v}$ and $y=uv \implies \frac{y}{x}=v^2$ and $xy=u^2$;
$\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix} v^{-1} & -uv^{-2} \\ v & u \end{vmatrix}=v^{-1}u+v^{-1}u=\frac{2u}{v}$;
$y=x \implies v=1$; $y=4x \implies v=2$; $xy=1 \implies u=1$; $xy=9 \implies u=3$.
Thus

$$\begin{aligned}\iint_R \left(\sqrt{\frac{y}{x}}+\sqrt{xy}\right)\,dx\,dy &= \int_1^3\int_1^2 (v+u)\left|-\frac{2u}{v}\right|\,dv\,du=\int_1^3\int_1^2\left(2u+\frac{2u^2}{v}\right)\,dv\,du\\ &=\int_1^3 \left[2uv+2u^2\ln v\right]_1^2\,du=\int_1^3 (2u+2u^3\ln 2)\,du\\ &=\left[u^2+\frac{2}{3}u^2\ln 2\right]_1^3=8+\frac{52}{3}\ln 2.\end{aligned}$$

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11. **Polar moment of inertia of an elliptical plate** A thin plate of constant density covers the region bounded by the ellipse $x^2/a^2+y^2/b^2=1$, $a>0$, $b>0$, in the xy-plane. Find the first moment of the plate about the origin. (*Hint:* Use the transformation $x=ar\cos\theta$, $y=br\sin\theta$.)

Solution. Let $x=ar\cos\theta$ and $y=ar\sin\theta$. Then

$$\frac{\partial(x,y)}{\partial(r,\theta)}=\begin{vmatrix} a\cos\theta & -ar\sin\theta \\ b\sin\theta & br\cos\theta \end{vmatrix}=abr\cos^2\theta+abr\sin^2\theta=abr.$$

Hence, the first moment of the plate about the origin is

$$\begin{aligned} I_0 &= \iint_R (x^2+y^2)\,dA=\int_0^{2\pi}\int_0^1 r^2(a^2\cos^2\theta+b^2\sin^2\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right|\,dr\,d\theta\\ &=\frac{ab}{4}\int_0^{2\pi}(a^2\cos^2\theta+b^2\sin^2\theta)\,d\theta\\ &=\frac{ab\pi(a^2+b^2)}{4}.\end{aligned}$$

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19. Evaluate

$$\iiint |xyz|\,dx\,dy\,dz$$

over the solid ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}\le 1.$$

(*Hint:* Let $x=au$, $y=bv$, and $z=cw$. Then integrate over an appropriate region in uvw-plane.)

Solution. The transformation $x = au$, $y = bv$, $z = cw$ satisfies

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc,$$

and takes the ellipsoid region $R : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1$ in xyz-space into the spherical region $G : u^2 + v^2 + w^2 \le 1$ in uvw-space. Hence,

$$\begin{aligned}
\iiint_R |xyz|\, dx\, dy\, dz &= \iiint_G |(au)(bv)(cw)||abc|\, du\, dv\, dw \\
&= 8a^2b^2c^2 \int_0^{\pi/2}\int_0^{\pi/2}\int_0^1 (\rho\cos\theta\sin\phi)(\rho\sin\theta\sin\phi)(\rho\cos\phi)\rho^2\sin\phi\, d\rho\, d\phi\, d\theta \\
&= \frac{4a^2b^2c^2}{3}\int_0^{\pi/2}\int_0^{\pi/2} \sin\theta\cos\theta\sin^3\phi\cos\phi\, d\phi\, d\theta \\
&= \frac{4a^2b^2c^2}{3}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/2}\left[\frac{\sin^4\phi}{4}\right]_0^{\pi/2} \\
&= \frac{a^2b^2c^2}{6}.
\end{aligned}$$

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22. Find the Jacobian $\partial(x,y,z)/\partial(u,v,w)$ of the transformation

(a) $x = u\cos v$, $y = u\sin v$, $z = w$.

(b) $x = 2u - 1$, $y = 3v - 4$, $z = (1/2)(w - 4)$.

Solution. (a) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \cos v & -u\sin v & 0 \\ \sin v & u\cos v & 0 \\ 0 & 0 & 1 \end{vmatrix} = (1)(u\cos^2 v + u\sin^2 v) = u.$$

(b) $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1/2 \end{vmatrix} = 3.$$

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23. Evaluate the appropriate determinant to show that the Jacobian of the transformation from Cartesian $\rho\phi\theta$-space to Cartesian xyz-space is $\rho^2\sin\phi$.

Solution.

$$\begin{aligned}
\frac{\partial(x,y,z)}{\partial(r,\phi,\theta)} &= \begin{vmatrix} \sin\phi\cos\theta & \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \\ \cos\phi & -\rho\sin\phi & 0 \end{vmatrix} \\
&= (\cos\phi)\begin{vmatrix} \rho\cos\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \rho\cos\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} + (\rho\sin\phi)\begin{vmatrix} \sin\phi\cos\theta & -\rho\sin\phi\sin\theta \\ \sin\phi\sin\theta & \rho\sin\phi\cos\theta \end{vmatrix} \\
&= (\rho^2\cos\theta)(\sin\phi\cos\phi\cos^2\theta + \sin\phi\cos\phi\sin^2\theta) + (\rho^2\sin\phi)(\sin^2\phi\cos^2\theta + \sin^2\phi\sin^2\theta) \\
&= \rho^2\sin\phi\cos^2\phi + \rho^2\sin^3\phi \\
&= \rho^2\sin\phi.
\end{aligned}$$

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