MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 3

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercise 15.6

4. **Finding a centroid** Find the centroid of the triangular region cut from the first quadrant by the line $x + y = 3$.

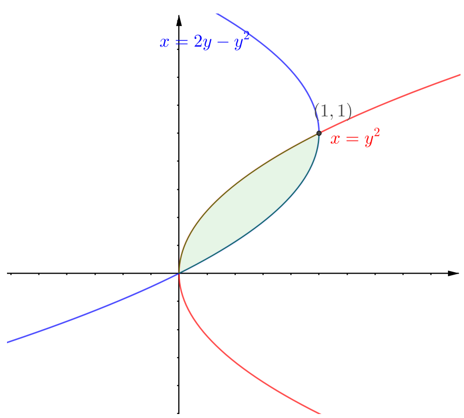
   Solution. $\mathbf{M} = \int_0^3 \int_0^{3-x} dy\,dx = \frac{9}{2}$;

   by symmetry, $\mathbf{M}_y = \mathbf{M}_x = \int_0^3 \int_0^{3-y} y\,dx\,dy = \frac{9}{2}$.

   Hence, the centroid is $(\bar{x}, \bar{y}) = (\mathbf{M}_y/\mathbf{M}, \mathbf{M}_x/\mathbf{M}) = (1, 1)$. ◀

14. **Finding a center of mass and moment of inertia** Find the center of mass and moment of inertia about the x-axis of a thin plate bounded by the curves $x = y^2$ and $x = 2y - y^2$ if the density at the point (x, y) is $\delta(x, y) = y + 1$.

    Solution.



$\mathbf{M} = \int_0^1 \int_{y^2}^{2y-y^2} (y+1)\,dx\,dy = \int_0^1 (2y - 2y^3)\,dy = \frac{1}{2}$;

$\mathbf{M}_x = \int_0^1 \int_{y^2}^{2y-y^2} y(y+1)\,dx\,dy = \int_0^1 (2y^2 - 2y^4)\,dy = \frac{4}{15}$;

$\mathbf{M}_y = \int_0^1 \int_{y^2}^{2y-y^2} x(y+1)\,dx\,dy = \int_0^1 (2y^2 - 2y^4)\,dy = \frac{4}{15}$.

Hence, the center of mass is $(\bar{x}, \bar{y}) = (\mathbf{M}_y/\mathbf{M}, \mathbf{M}_x/\mathbf{M}) = (\frac{8}{15}, \frac{8}{15})$.

The moment of inertia about the x-axis is

$$I_x = \int_0^1 \int_{y^2}^{2y-y^2} y^2(y+1)\,dx\,dy = 2\int_0^1 (y^3 - y^5)\,dy = \frac{1}{6}.$$

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25. (a) **Center of mass** Find the center of mass of a solid of constant density bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 4$.
(b) Find the plane $z = c$ that divides the solid into two parts of equal volume. This plane does not pass through the center of mass.

Solution. (a) $\mathbf{M} = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 dz\,dy\,dz = 4\int_0^{\pi/2} \int_0^2 \int_{r^2}^4 r\,dz\,dr\,d\theta = 8\pi$;
$\mathbf{M}_{xy} = 4\int_0^2 \int_0^{\sqrt{4-x^2}} \int_{x^2+y^2}^4 z\,dz\,dy\,dz = 4\int_0^{\pi/2} \int_0^2 \int_{r^2}^4 zr\,dz\,dr\,d\theta = \frac{64\pi}{3}$.
So $\bar{z} = \mathbf{M}_{xy}/\mathbf{M} = \frac{8}{3}$, and $\bar{x} = \bar{y} = 0$ by symmetry. Hence, the center of mass of the solid is $(\bar{x}, \bar{y}, \bar{z}) = (0, 0, \frac{8}{3})$.
(b)

$$\begin{aligned}
4\pi &= 4\int_0^{\pi/2} \int_0^{\sqrt{c}} \int_{r^2}^c r\,dz\,dr\,d\theta = 4\int_0^{\pi/2} \int_0^{\sqrt{c}} (cr - r^3)\,dr\,d\theta \\
\pi &= \int_0^{\pi/2} \frac{c^2}{4}\,d\theta = \frac{c^2\pi}{8} \\
c^2 &= 8 \\
c &= 2\sqrt{2} \qquad \text{since } c > 0.
\end{aligned}$$

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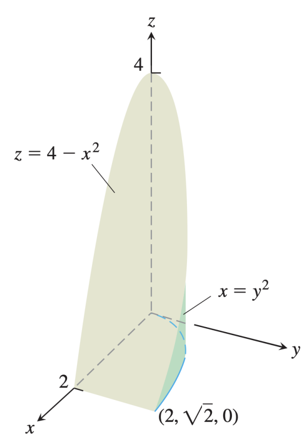
30. Find (a) the mass (b) the center of mass of the following solid. A solid in the first octant is bounded by the planes $y = 0$ and $z = 0$ and by the surfaces $z = 4 - x^2$ and $x = y^2$ (see the accompanying figure). Its density function is $\delta(x, y, z) = kxy$, k is a constant.



Solution. (a) The mass is $\mathbf{M} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} kxy\,dz\,dy\,dx = k\int_0^2 \int_0^{\sqrt{x}} xy(4-x^2)\,dy\,dx = \frac{k}{2}\int_0^2 (4x^2 - x^4)\,dx = \frac{32k}{15}$.

(b) $\mathbf{M}_{yz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} x(kxy)\,dz\,dy\,dx = \frac{k}{2}\int_0^2 (4x^3 - x^5)\,dx = \frac{8k}{3}$;
$\mathbf{M}_{xz} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} y(kxy)\,dz\,dy\,dx = \frac{k}{3}\int_0^2 (4x^{5/2} - x^{9/2})\,dx = \frac{256\sqrt{2}k}{231}$;
$\mathbf{M}_{xy} = \int_0^2 \int_0^{\sqrt{x}} \int_0^{4-x^2} z(kxy)\,dz\,dy\,dx = \frac{k}{4}\int_0^2 x^2(4 - x^2)^2\,dx = \frac{256k}{105}$.

Hence, the center of mass is $(\bar{x}, \bar{y}, \bar{z}) = (\mathbf{M}_{yz}/\mathbf{M}, \mathbf{M}_{xz}/\mathbf{M}, \mathbf{M}_{xy}/\mathbf{M}) = (\frac{5}{4}, \frac{40\sqrt{2}}{77}, \frac{8}{7})$. ◀

Exercise 15.7

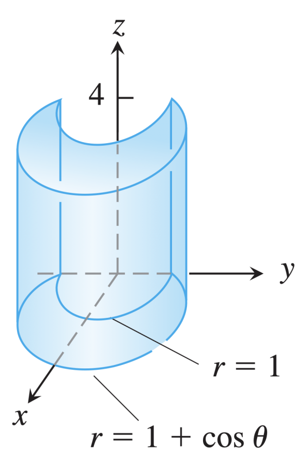
6. Evaluate the cylindrical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}(r^2\sin^2\theta + z^2)\,dz\,rdr\,d\theta$.

Solution. $\displaystyle\int_0^{2\pi}\int_0^1\int_{-1/2}^{1/2}\left(r^2\sin^2\theta + z^2\right)\,dz\,rdr\,d\theta = \int_0^{2\pi}\int_0^1(r^3\sin^2\theta + \frac{r}{12})dr\,d\theta$
$\displaystyle = \int_0^{2\pi}\left(\frac{\sin^2\theta}{4} + \frac{1}{24}\right)\,d\theta = \frac{\pi}{3}$. ◀

9. Evaluate the integral $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}(r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz$.

Solution. $\displaystyle\int_0^1\int_0^{\sqrt{z}}\int_0^{2\pi}(r^2\cos^2\theta + z^2)\,r\,d\theta\,dr\,dz = \int_0^1\int_0^{\sqrt{z}}(\pi r^3 + 2\pi r z^2)\,dr\,dz$
$\displaystyle = \int_0^1\left(\frac{\pi z^2}{4} + \pi z^3\right)\,dz = \frac{\pi}{3}$. ◀

17. Set up the iterated integral for evaluating $\displaystyle\iiint_D f(r,\theta,z)\,dz\,rdr\,d\theta$ over the given region D, where D is the solid right cylinder whose base is the region in the xy-plane that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ and whose top lies in the plane $z = 4$.



Solution. $\displaystyle\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta}\int_0^4 f(r,\theta,z)\,dz\,rdr\,d\theta$ ◀

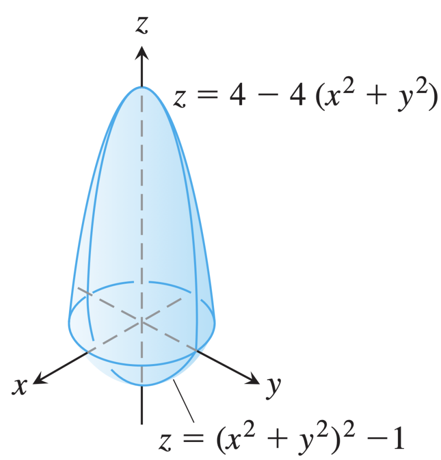
23. Evaluate the spherical coordinate integral $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$.

Solution. $\displaystyle\int_0^{2\pi}\int_0^{\pi}\int_0^{(1-\cos\phi)/2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \frac{1}{24}\int_0^{2\pi}\int_0^{\pi}(1-\cos\phi)^3\sin\phi\,d\phi\,d\theta$
$\displaystyle = \frac{1}{96}\int_0^{2\pi}\left[(1-\cos\phi)^4\right]_0^{\pi}\,d\theta = \frac{2\pi}{96}[2^4-0] = \frac{\pi}{3}.$ ◀

30. Evaluate the integral $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2}5\rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi$.

Solution. $\displaystyle\int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}\int_{\csc\phi}^{2}5\rho^4\sin^3\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32-\csc^5\phi)\sin^3\phi\,d\theta\,d\phi$
$\displaystyle = \int_{\pi/6}^{\pi/2}\int_{-\pi/2}^{\pi/2}(32\sin^3\phi-\csc^2\phi)\,d\theta\,d\phi = \pi\left\{\int_{\pi/6}^{\pi/2}-32(1-\cos^2\phi)\,d(\cos\phi)+\Big[\cot\phi\Big]_{\pi/6}^{\pi/2}\right\}$
$\displaystyle = \pi\left\{-32\left[\cos\phi-\frac{\cos^3\phi}{3}\right]_{\pi/6}^{\pi/2}+[0-\sqrt{3}]\right\} = \pi\left\{-32[-\frac{\sqrt{3}}{2}+\frac{1}{3}(\frac{\sqrt{3}}{2})^3]-\sqrt{3}\right\} = 11\pi\sqrt{3}.$ ◀

43. Find the volume of the solid.



Solution. Using cylindrical coordinates,

$$\text{Volume} = \int_0^{2\pi}\int_0^{1}\int_{r^4-1}^{4-4r^2}dz\,r dr\,d\theta = \int_0^{2\pi}\int_0^{1}(5r-4r^3-r^5)\,dr\,d\theta$$
$$= \int_0^{2\pi}\left(\frac{5}{2}-1-\frac{1}{6}\right)\,d\theta = \frac{8\pi}{3}.$$

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