MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 2

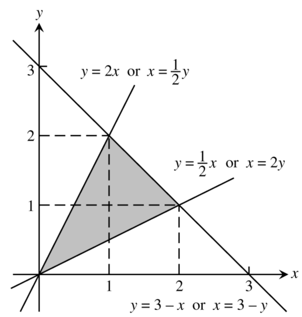
Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercise 15.3

11. Sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

    The lines $y = 2x$, $y = x/2$ and $y = 3 - x$.

    Solution.



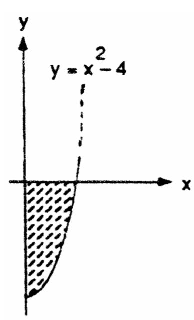
$$\text{Area} = \int_0^1 \int_{x/2}^{2x} dy\,dx + \int_1^2 \int_{x/2}^{3-x} dy\,dx = \int_0^1 [y]_{x/2}^{2x}\,dx + \int_1^2 [y]_{x/2}^{3-x}\,dx$$

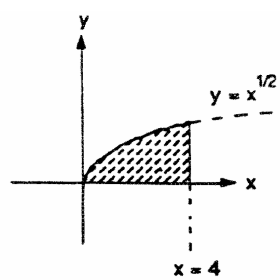
$$= \int_0^1 (\frac{3}{2}x)\,dx + \int_1^2 (3 - \frac{3}{2}x)\,dx = \left[\frac{3}{4}x^2\right]_0^1 + \left[3x - \frac{3}{4}x^2\right]_1^2 = \frac{3}{2}.$$ ◀

18. The integrals and sum of integrals give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

    $$\int_0^2 \int_{x^2-4}^{0} dy\,dx + \int_0^4 \int_0^{\sqrt{x}} dy\,dx$$

    Solution.





$$\int_0^2 \int_{x^2-4}^{0} dy\,dx + \int_0^4 \int_0^{\sqrt{x}} dy\,dx = \int_0^2 (4 - x^2)\,dx + \int_0^4 x^{1/2}\,dx$$

$$= \left[4x - \frac{x^3}{3}\right]_0^2 + \left[\frac{2}{3}x^{3/2}\right]_0^4 = \frac{32}{3}.$$ ◀

Exercise 15.4

18. Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx$$

Solution. $\displaystyle\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{2}{(1+x^2+y^2)^2}\,dy\,dx = 4\int_{0}^{\pi/2}\int_{0}^{1} \frac{2}{(1+r^2)^2}\cdot r\,dr\,d\theta$
$\displaystyle = 4\int_{0}^{\pi/2}\left[-\frac{1}{1+r^2}\right]_0^1 d\theta = 2\int_{0}^{\pi/2} d\theta = \pi..$ ◀

27. Find the area of the region cut from the first quadrant by the curve $r = 2(2-\sin 2\theta)^{1/2}$.

Solution. Area $\displaystyle = \int_{0}^{\pi/2}\int_{0}^{2(2-\sin 2\theta)^{1/2}} r\,dr\,d\theta = 2\int_{0}^{\pi/2}(2-\sin 2\theta)\,d\theta = 2(\pi-1).$ ◀

30. **Snail Shell** Find the area of the region enclosed by the positive x-axis and spiral $r = 4\theta/3$, $0 \le \theta \le 2\pi$. The region looks like a snail shell.

Solution. Area $\displaystyle = \int_{0}^{2\pi}\int_{0}^{4\theta/3} r\,dr\,d\theta = \frac{8}{9}\int_{0}^{2\pi}\theta^2\,d\theta = \frac{64\pi^3}{27}.$ ◀

35. **Average distance from interior of disk to center** Find the average distance from a point $P(x,y)$ in the disk $x^2+y^2 \le a^2$ to the origin.

Solution. Average distance $\displaystyle = \frac{1}{\pi a^2}\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \frac{1}{\pi a^2}\int_{0}^{2\pi}\int_{0}^{a} r\cdot r\,dr\,d\theta$
$\displaystyle = \frac{1}{\pi a^2}\int_{0}^{2\pi}\frac{a^3}{3}\,d\theta = \frac{2a}{3}.$ ◀

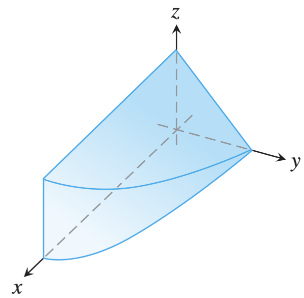
Exercise 15.5

18. Evaluate the integral

$$\int_{0}^{1}\int_{1}^{\sqrt{e}}\int_{1}^{e} se^s \ln r \,\frac{(\ln t)^2}{t}\,dt\,dr\,ds \quad (rst\text{-space})$$

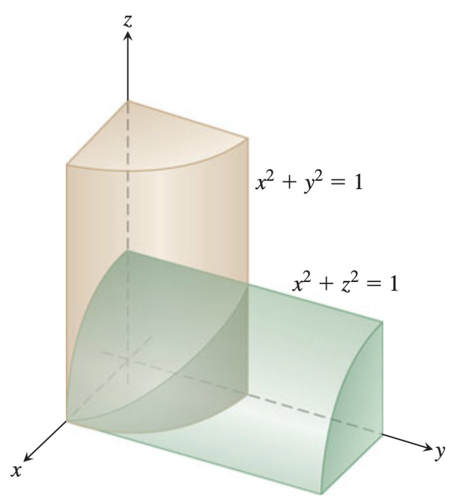
Solution. $\displaystyle\int_{0}^{1}\int_{1}^{\sqrt{e}}\int_{1}^{e} se^s \ln r \,\frac{(\ln t)^2}{t}\,dt\,dr\,ds = \int_{0}^{1}\int_{1}^{\sqrt{e}} (se^s \ln r)\left[\frac{(\ln t)^3}{3}\right]_1^e dt\,dr\,ds$
$\displaystyle = \int_{0}^{1}\int_{1}^{\sqrt{e}} \frac{1}{3}se^s \ln r\,dr\,ds = \int_{0}^{1}\frac{se^s}{3}\left[r\ln r - r\right]_1^{\sqrt{e}} ds = \frac{2-\sqrt{e}}{6}\int_{0}^{1} se^s\,ds$
$\displaystyle = \frac{2-\sqrt{e}}{6}\left[se^s - e^s\right]_0^1 = \frac{2-\sqrt{e}}{6}.$ ◀

25. Find the volume of the region: The region in the first octant bounded by the coordinate planes, the plane $y + z = 2$, and the cylinder $x = 4 - y^2$.



Solution. Volume $= \int_0^4 \int_0^{\sqrt{4-x}} \int_0^{2-y} dz\, dy\, dx = \int_0^4 \int_0^{\sqrt{4-x}} (2-y)\, dy\, dx$
$= \int_0^4 \left[2\sqrt{4-x} - \frac{1}{2}(4-x) \right] dx = \left[-\frac{4}{3}(4-x)^{3/2} + \frac{1}{4}(4-x)^2 \right]_0^4 = \frac{20}{3}.$ ◀

29. Find the volume of the region: The region common to the interior of the cylinders $x^2+y^2 = 1$ and $x^2 + z^2 = 1$, one-eighth of which is show in the accompanying figure



Solution. Volume $= 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2}} dz\, dy\, dx = 8 \int_0^1 \int_0^{\sqrt{1-x^2}} \sqrt{1-x^2}\, dy\, dx$
$= 8 \int_0^1 (1-x^2)\, dx = \frac{16}{3}.$ ◀

42. Evaluate the integral by changing the order of integration in an appropriate way.

$$\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\, dy\, dx\, dz.$$

Solution. $\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\, dy\, dx\, dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2}\, dx\, dy\, dz$
$= \int_0^1 \int_0^1 6yze^{zy^2}\, dy\, dz = \int_0^1 \left[3e^{zy^2} \right]_0^1 dz = 3 \int_0^1 (e^z - z)\, dz = 3[e^z - 1]_0^1 = 3e - 6.$ ◀