MATH 2020B Advanced Calculus II
2023-24 Term 2
Suggested Solution of Homework 10

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.8

5. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

   Cube $\quad \mathbf{F} = (y-x)\mathbf{i} + (z-y)\mathbf{j} + (y-x)\mathbf{k}$

   D: The cube bounded by the planes $x = \pm 1$, $y = \pm 1$, and $z = \pm 1$.

   Solution. $\nabla \cdot \mathbf{F} = -2$. By Divergence Theorem,

$$\text{Flux} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} -2\, dx\, dy\, dz = -2(2^3) = -16.$$

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8. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

   Sphere $\quad \mathbf{F} = x^2\mathbf{i} + xz\mathbf{j} + 3z\mathbf{k}$

   D: The solid sphere $x^2 + y^2 + z^2 \leq 4$.

   Solution. $\nabla \cdot \mathbf{F} = 2x + 3$. By Divergence Theorem,

$$\begin{aligned}\text{Flux} &= \iiint_D \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi}\int_0^{\pi}\int_0^{2} (2\rho\sin\phi\cos\theta + 3)(\rho^2\sin\phi)\, d\rho\, d\phi\, d\theta \\ &= 32\pi.\end{aligned}$$

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10. **Cylindrical can** $\quad \mathbf{F} = (6x^2 + 2xy)\mathbf{i} + (2y + x^2 z)\mathbf{j} + 4x^2y^3\mathbf{k}$

    D: The region cut from the first octant by the cylinder $x^2 + y^2 = 4$ and the plane $z = 3$.

    Solution. $\nabla \cdot \mathbf{F} = 12x + 2y + 2$. By Divergence Theorem,

$$\begin{aligned}\text{Flux} &= \iiint_D \nabla \cdot \mathbf{F}\, dV = \int_0^{3}\int_0^{\pi/2}\int_0^{2} (12r\cos\theta + 2r\sin\theta + 2)\, r dr\, d\theta\, dz \\ &= 112 + 6\pi.\end{aligned}$$

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15. **Thick sphere** $\mathbf{F} = (5x^3 + 12xy^2)\mathbf{i} + (y^3 + e^y \sin z)\mathbf{j}$
D: The solid region between the spheres $x^2 + y^2 + z^2 = 1$ and $x^2 + y^2 + z^2 = 2$.

Solution. $\nabla \cdot \mathbf{F} = 15x^2 + 15y^2 + 15z^2 = 15\rho^2$. By Divergence Theorem,

$$\begin{aligned}\text{Flux} &= \iiint_D \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_1^{\sqrt{2}} (15\rho^2)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta \\ &= \left(48\sqrt{2} - 12\right)\pi.\end{aligned}$$

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19. Let $\mathbf{F} = (y\cos 2x)\mathbf{i} + (y^2 \sin 2x)\mathbf{j} + (x^2 y + z)\mathbf{k}$. Is there a vector field $\mathbf{A}$ such that $\mathbf{F} = \nabla \times \mathbf{A}$? Explain your answer.

Solution. Suppose $\mathbf{A}$ is a field such that $\nabla \times \mathbf{A} = \mathbf{F}$. Then

$$\nabla \cdot (\nabla \times \mathbf{A}) = 0$$

but

$$\nabla \cdot \mathbf{F} = (-2y \sin 2x) + (2y \sin 2x) + 1 = 1 \neq 0.$$

Contradiction. So no such field $\mathbf{A}$ exists. ◀

20. **Outward flux of a gradient field** Let S be the surface of the portion of the solid sphere $x^2 + y^2 + z^2 \leq a^2$ that lies in the first octant and let $f(x, y, z) = \ln\sqrt{x^2 + y^2 + z^2}$. Calculate

$$\iint_S \nabla f \cdot \mathbf{n}\, d\sigma.$$

($\nabla f \cdot \mathbf{n}$ is the derivative of f in the direction of outward normal $\mathbf{n}$.)

Solution. From the Divergence Theorem,

$$\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot (\nabla f)\, dV = \iiint_D \left(\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}\right) dV.$$

Now,

$$\frac{\partial f}{\partial x} = \frac{x}{x^2 + y^2 + z^2}, \qquad \frac{\partial^2 f}{\partial x^2} = \frac{-x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2},$$

and so,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} = \frac{x^2 + y^2 + z^2}{(x^2 + y^2 + z^2)^2} = \frac{1}{x^2 + y^2 + z^2}.$$

Therefore,

$$\iint_S \nabla f \cdot \mathbf{n}\, d\sigma = \iiint_D \frac{1}{x^2 + y^2 + z^2}\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^a \frac{\rho^2 \sin\phi}{\rho^2}\, d\rho\, d\phi\, d\theta = \frac{\pi a}{2}.$$

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23. Calculate the net outward flux of the vector field

$$\mathbf{F} = xy\mathbf{i} + (\sin xz + y^2)\mathbf{j} + (e^{xy^2} + z)\mathbf{k}$$

over the surface S surrounding the region D bounded by the planes $y = 0$, $z = 0$, $z = 2 - y$ and the parabolic cylinder $z = 1 - x^2$.

Solution. $\nabla \cdot \mathbf{F} = y + 2y + 1 = 3y + 1$. By Divergence Theorem,

$$\begin{aligned}
\text{Flux} = \iiint_D \nabla \cdot \mathbf{F}\, dV &= \int_{-1}^{1} \int_{0}^{1-x^2} \int_{0}^{2-z} (3y+1)\, dy\, dz\, dx \\
&= \int_{-1}^{1} \int_{0}^{1-x^2} \left(\frac{3}{2}(2-z)^2 + (2-z) \right) dz\, dx \\
&= \int_{-1}^{1} \left(-\frac{(1+x^2)^3}{2} - \frac{(1+x^2)^2}{2} + 6 \right) dx \\
&= \frac{776}{105}.
\end{aligned}$$

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25. Let $\mathbf{F}$ be a differentiable vector field and let $g(x, y, z)$ be a differentiable scalar function. Verify the following identities.

    (a) $\nabla \cdot (g\mathbf{F}) = g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}$

    (b) $\nabla \times (g\mathbf{F}) = g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}$

Solution. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$.

(a)

$$\begin{aligned}
\nabla \cdot (g\mathbf{F}) &= \frac{\partial}{\partial x}(gM) + \frac{\partial}{\partial y}(gN) + \frac{\partial}{\partial z}(gP) \\
&= \left(g\frac{\partial M}{\partial x} + M\frac{\partial g}{\partial x} \right) + \left(g\frac{\partial N}{\partial y} + N\frac{\partial g}{\partial y} \right) + \left(g\frac{\partial P}{\partial z} + P\frac{\partial g}{\partial z} \right) \\
&= g\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z} \right) + \left(M\frac{\partial g}{\partial x} + N\frac{\partial g}{\partial y} + P\frac{\partial g}{\partial z} \right) \\
&= g\nabla \cdot \mathbf{F} + \nabla g \cdot \mathbf{F}
\end{aligned}$$

(b)

$$\begin{aligned}
\nabla \cdot (g\mathbf{F}) &= \left[\frac{\partial}{\partial y}(gP) - \frac{\partial}{\partial z}(gN) \right] \mathbf{i} + \left[\frac{\partial}{\partial z}(gM) - \frac{\partial}{\partial x}(gP) \right] \mathbf{j} + \left[\frac{\partial}{\partial x}(gN) - \frac{\partial}{\partial y}(gM) \right] \mathbf{k} \\
&= \left(g\frac{\partial P}{\partial y} - g\frac{\partial N}{\partial z} \right) \mathbf{i} + \left(P\frac{\partial g}{\partial y} - N\frac{\partial g}{\partial z} \right) \mathbf{i} + \left(g\frac{\partial M}{\partial z} - g\frac{\partial P}{\partial x} \right) \mathbf{j} + \left(M\frac{\partial g}{\partial z} - P\frac{\partial g}{\partial x} \right) \mathbf{j} \\
&\quad + \left(g\frac{\partial N}{\partial x} - g\frac{\partial M}{\partial y} \right) \mathbf{k} + \left(N\frac{\partial g}{\partial x} - M\frac{\partial g}{\partial y} \right) \mathbf{k} \\
&= g\nabla \times \mathbf{F} + \nabla g \times \mathbf{F}
\end{aligned}$$

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26. Let $\mathbf{F}_1$ and $\mathbf{F}_2$ be differentiable vector fields and let a and b be arbitrary real constants. Verify the following identities.

    (a) $\nabla \cdot (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \cdot \mathbf{F}_1 + b\nabla \cdot \mathbf{F}_2$

    (b) $\nabla \times (a\mathbf{F}_1 + b\mathbf{F}_2) = a\nabla \times \mathbf{F}_1 + b\nabla \times \mathbf{F}_2$

    (c) $\nabla \cdot (\mathbf{F}_1 \times \mathbf{F}_2) = \mathbf{F}_2 \cdot \nabla \times \mathbf{F}_1 - \mathbf{F}_1 \cdot \nabla \times \mathbf{F}_2$

Solution. Only (c) is provided since (a) and (b) are straightforward.

Let $\mathbf{F}_1 = M_1\mathbf{i} + N_1\mathbf{j} + P_1\mathbf{k}$ and $\mathbf{F}_2 = M_2\mathbf{i} + N_2\mathbf{j} + P_2\mathbf{k}$. Then

$$\mathbf{F}_1 \times \mathbf{F}_2 = (N_1P_2 - P_1N_2)\mathbf{i} - (M_1P_2 - P_1M_2)\mathbf{j} + (M_1N_2 - N_1M_2)\mathbf{k},$$

and so

$$\begin{aligned}\nabla\cdot(\mathbf{F}_1\times\mathbf{F}_2) &= \left(P_2\frac{\partial N_1}{\partial x} + N_1\frac{\partial P_2}{\partial x} - N_2\frac{\partial P_1}{\partial x} - P_1\frac{\partial N_2}{\partial x}\right) - \left(P_2\frac{\partial M_1}{\partial y} + M_1\frac{\partial P_2}{\partial y} - M_2\frac{\partial P_1}{\partial y} - P_1\frac{\partial M_2}{\partial y}\right)\\
&\quad + \left(N_2\frac{\partial M_1}{\partial z} + M_1\frac{\partial N_2}{\partial z} - M_2\frac{\partial N_1}{\partial z} - N_1\frac{\partial M_2}{\partial z}\right)\\
&= M_2\left(\frac{\partial P_1}{\partial y} - \frac{\partial N_1}{\partial z}\right) + N_2\left(\frac{\partial M_1}{\partial z} - \frac{\partial P_1}{\partial x}\right) + P_2\left(\frac{\partial N_1}{\partial x} - \frac{\partial M_1}{\partial y}\right)\\
&\quad - M_1\left(\frac{\partial P_2}{\partial y} - \frac{\partial N_2}{\partial z}\right) - N_1\left(\frac{\partial M_2}{\partial z} - \frac{\partial P_2}{\partial x}\right) - P_1\left(\frac{\partial N_2}{\partial x} - \frac{\partial M_2}{\partial y}\right)\\
&= \mathbf{F}_2\cdot\nabla\times\mathbf{F}_1 - \mathbf{F}_1\cdot\nabla\times\mathbf{F}_2.\end{aligned}$$

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29. **Green's first formula** Suppose that f and g are scalar functions with continuous first- and second-order partial derivatives through- out a region D that is bounded by a closed piecewise smooth surface S. Show that

$$\iint_S f\nabla g\cdot\mathbf{n}\,d\sigma = \iiint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV. \tag{10}$$

Equation (10) is **Green's first formula**. (Hint: Apply the Divergence Theorem to the field $\mathbf{F} = f\nabla g$.)

Solution. By the Divergence Theorem,

$$\begin{aligned}\iint_S f\nabla g\cdot\mathbf{n}\,d\sigma &= \iiint_D \nabla\cdot(f\nabla g)\,dV\\
&= \iiint_D \left(f\frac{\partial^2 g}{\partial x^2} + \frac{\partial f}{\partial x}\frac{\partial g}{\partial x} + f\frac{\partial^2 g}{\partial y^2} + \frac{\partial f}{\partial y}\frac{\partial g}{\partial y} + f\frac{\partial^2 g}{\partial z^2} + \frac{\partial f}{\partial z}\frac{\partial g}{\partial z}\right)dV\\
&= \iiint_D (f\nabla^2 g + \nabla f\cdot\nabla g)\,dV.\end{aligned}$$

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