(Cont'd)

To see the relation between $\vec{F}\cdot\hat{n}\,d\sigma$ and ζ,

we parametrize $S = \partial D$:

$$\vec{r}(u,v) = x(u,v)\hat{i} + y(u,v)\hat{j} + z(u,v)\hat{k}$$

$$\begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix}\hat{i} + \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix}\hat{j} + \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix}\hat{k}$$

$$= \frac{\partial(y,z)}{\partial(u,v)}\hat{i} + \frac{\partial(z,x)}{\partial(u,v)}\hat{j} + \frac{\partial(x,y)}{\partial(u,v)}\hat{k}$$

If $\vec{r}_u\times\vec{r}_v$ is <u>outward</u>, then

$$\hat{n} = \frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|} \quad \text{and} \quad d\sigma = \|\vec{r}_u\times\vec{r}_v\|\,du\,dv$$

$$= \|\vec{r}_u\times\vec{r}_v\|\,du\wedge dv \quad \begin{pmatrix}\text{correct} \\ \text{orientation}\end{pmatrix}$$

Then $\vec{F}\cdot\hat{n}\,d\sigma = \vec{F}\cdot(\vec{r}_u\times\vec{r}_v)\,du\wedge dv$

$$= \zeta_1 \frac{\partial(y,z)}{\partial(u,v)}du\wedge dv + \zeta_2\frac{\partial(z,x)}{\partial(u,v)}du\wedge dv + \zeta_3\frac{\partial(x,y)}{\partial(u,v)}du\wedge dv$$

$$= \zeta_1\, dy\wedge dz + \zeta_2\, dz\wedge dx + \zeta_3\, dx\wedge dy = \zeta$$

Hence divergence thm is

$$\boxed{\iiint_D d\zeta = \iint_{\partial D} \zeta} \qquad \zeta = 2\text{-form}$$

eg3 Stokes' Thm

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \longleftrightarrow \omega = M\,dx + N\,dy + L\,dz$$

Then $d\omega = dM \wedge dx + dN \wedge dy + dL \wedge dz$

$$= M_y\, dy \wedge dx + M_z\, dz \wedge dx$$
$$+ N_x\, dx \wedge dy + N_z\, dz \wedge dy$$
$$+ L_x\, dx \wedge dz + L_y\, dy \wedge dz$$

$$= (L_y - N_z)\, dy \wedge dz + (M_z - L_x)\, dz \wedge dx + (N_x - M_y)\, dx \wedge dy$$

As in eg 2, $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ & $d\sigma = \|\vec{r}_u \times \vec{r}_v\|\, du \wedge dv$

$$\vec{\nabla} \times \vec{F} = (L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k} \quad \text{(Check!)}$$

$$\Rightarrow (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma = (\vec{\nabla} \times \vec{F}) \cdot (\vec{r}_u \times \vec{r}_v)\, du \wedge dv$$

$$= (L_y - N_z)\frac{\partial(y,z)}{\partial(u,v)}\, du \wedge dv + (M_z - L_x)\frac{\partial(z,x)}{\partial(u,v)}\, du \wedge dv$$
$$+ (N_x - M_y)\frac{\partial(x,y)}{\partial(u,v)}\, du \wedge dv$$

$$= (L_y - N_z)\, dy \wedge dz + (M_z - L_x)\, dz \wedge dx + (N_x - M_y)\, dx \wedge dy$$

$$= d\omega$$

Stoke's Thm becomes

$$\oint_{C = \partial S} \vec{F} \cdot d\vec{r} \longrightarrow \boxed{\oint_{\partial S} \omega = \iint_S d\omega} \longleftarrow \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

Generalization to manifold of n-dimension with boundary

(Skipped)

- M = n dim'l Manifold (oriented)
- ∂M = boundary (oriented with "induced" orientation)
- ω = $(n-1)$-form on M (smooth)

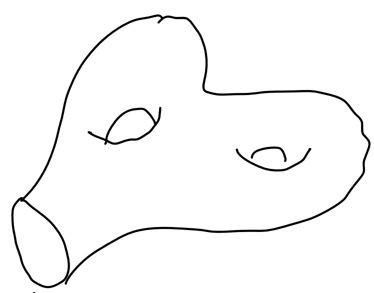
Then
$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

n-dim'l integral   (n-1)-dim'l integral

<u>Note</u>: ∂M is always <u>closed</u>, i.e. no boundary.

$$\therefore \quad \boxed{\partial(\partial M) = \partial^2 M = 0}$$

<u>boundary has no boundary</u>



∂S is a closed curve

Hence if $\omega = d\eta$, for some $(n-2)$-form η,

then $\int_M d(d\eta) = \int_M d\omega = \int_{\partial M} \omega$

$= \int_{\partial M} d\eta = \int_{\partial(\partial M)} \eta = 0$ (for any η.)

This suggests $\boxed{d^2\eta = 0}$, $\forall$ differential form η

Ex: Verify this for 0-form and 1-form in $\mathbb{R}^3$ and observes that these are just

$$\begin{cases} \vec{\nabla} \times \vec{\nabla} f = \vec{0} & (d^2 f = 0) \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 & (d^2\omega = 0) \end{cases}$$

eg: Let $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

check: $d\omega = 0$

But $\omega \neq df$ for any smooth function on $\mathbb{R}^2 \setminus \{(0,0)\}$

(since $\omega = d\theta$ and θ is not defined on $\mathbb{R}^2 \setminus \{(0,0)\}$)

Hence $d\omega = 0 \not\Rightarrow \omega = d\eta$ in general

$(\Longleftarrow)$

Note: Thm 10 can be written as:

$$\boxed{\begin{array}{l} \Omega \subset \mathbb{R}^2 \text{ simply-connected, then} \\ d\omega = 0 \Leftrightarrow \omega = df \quad \text{for some smooth function } f \text{ on } \Omega \end{array}}$$

Review

Double integrals

- Riemann sum, integrability, Fubini's Thm,
- Polar coordinates, improper integrals
- Applications: area, average, etc.

Triple integrals

- Riemann sum, integrability, Fubini's Thm,
- cylindrical & spherical coordinates, improper integrals
- Applications: volume, average, etc

Change of Variables

- Chain Rule, Jacobian (determinant)

mid-term

Vecta Analysis

- Vecta fields, gradient of a function ($\vec{\nabla} f$)
- line integral of functions, arc-length
- line integral of vecta fields: flow & flux
- simple-closed curves, orientation of curves
- Conservative vecta fields (Thms 8, 9 & 10)
- Simply-connected domains
- Curl & Div ($\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \cdot \vec{F}$)

- Surface integrals, area elements, orientation,
- Surface integrals of vector fields (flux)
- Green's, Stokes' & Divergence Thm
- Differential Forms

<u>Final Exam</u> May 3 (Fri) 9:30 – 11:30 U Gym

<u>Coverage</u>:
- All material in lecture notes, tutorial notes, textbook (Ch 15, 16) & homework assignments,
- except differential forms
- emphasis on those material not included in Midterm.
- 5 questions, answer all. Some are unfamiliar/difficult questions as required by the grade descriptor of A range,

(Note: Textbook & assignments contain <u>only</u> basic theory and basic questions.)

(End)