Change of Variables Formula : ($\mathbb{R}^2$)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\Rightarrow \begin{cases} dx = x_u\,du + x_v\,dv \\ dy = y_u\,du + y_v\,dv \end{cases}$$

$$\Rightarrow dx \wedge dy = (x_u\,du + x_v\,dv) \wedge (y_u\,du + y_v\,dv)$$

$$= (x_u y_v - x_v y_u)\,du \wedge dv$$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du \wedge dv$$

$$= \frac{\partial(x,y)}{\partial(u,v)}\,du \wedge dv$$

$$\boxed{dx \wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\,du \wedge dv}$$

Jacobian determinant.

Hence naturally

$$\boxed{\iint f(x,y)\,dx \wedge dy = \iint f(x(u,v), y(u,v))\,\frac{\partial(x,y)}{\partial(u,v)}\,du \wedge dv}$$

Compare with

$$\boxed{\iint f(x,y)\,dx\,dy = \iint f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv}$$

Similarly for $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

Then $$\boxed{dx\wedge dy\wedge dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du\wedge dv\wedge dw}$$

Pf: $$dx\wedge dy\wedge dz = (x_u du + x_v dv + x_w dw)\wedge(y_u du + y_v dv + y_w dw) \wedge (z_u du + z_v dv + z_w dw)$$

$$= (x_u y_v z_w - x_u y_w z_v)\, du\wedge dv\wedge dw$$
$$+ (-x_v y_u z_w + x_v y_w z_u)\, du\wedge dv\wedge dw$$
$$+ (x_w y_u z_v - x_w y_v z_u)\, du\wedge dv\wedge dw \quad \text{(Ex!)}$$

$$= [x_u(y_v z_w - y_w z_v) - x_v(y_u z_w - y_w z_u) + x_w(y_u z_v - y_v z_u)]\, du\wedge dv\wedge dw$$

$$= \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} du\wedge dv\wedge dw$$

✗

- "Oriented" change of variables formula
- "$dx\wedge dy$" oriented area element
- "$dx\wedge dy\wedge dz$" oriented volume element.

(see later remark)

Exterior differentiation "d" on a form "ω".

(0-form) f	df (1-form)
(1-form) $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$	$d\omega = d\omega_1 \wedge dx + d\omega_2 \wedge dy + d\omega_3 \wedge dz$ (2-form)
$\zeta = \zeta_1 dy \wedge dz + \zeta_2 dz \wedge dx + \zeta_3 dx \wedge dy$ (2-form)	$d\zeta = d\zeta_1 \wedge dy \wedge dz + d\zeta_2 \wedge dz \wedge dx + d\zeta_3 \wedge dx \wedge dy$ (3-form)
(3-form) $f\, dx \wedge dy \wedge dz$	$df \wedge dx \wedge dy \wedge dz = 0$ (4-form) in $\mathbb{R}^3$

eg 0 : $d(dx) = d(dy) = d(dz) = 0$

eg 1 (in $\mathbb{R}^2$) $\omega = M\,dx + N\,dy$ $\quad (M = M(x,y),\ N = N(x,y))$

then
$$\begin{aligned} d\omega &= dM \wedge dx + dN \wedge dy \\ &= (M_x dx + M_y dy) \wedge dx + (N_x dx + N_y dy) \wedge dy \\ &= (N_x - M_y)\, dx \wedge dy \end{aligned}$$

(+ve) oriented area element

In this notation, Green's Thm

$$\oint_{C = \partial R} M dx + N dy = \iint_R (N_x - M_y)\, dx\, dy$$

can be written as

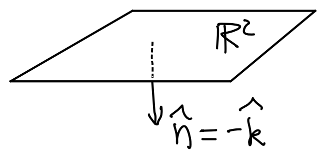
$$\boxed{\oint_{\partial R} \omega = \iint_R d\omega}$$

<u>Remark</u> : If we let $\vec{F} = M\hat{i} + N\hat{j} \longleftrightarrow \omega = M\,dx + N\,dy$

then $(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,dA = (N_x - M_y)\underbrace{\hat{k}\cdot\hat{n}\,dA}_{dx\wedge dy} \longleftrightarrow d\omega$

$(\hat{n}=\hat{k})$

and if we use



,

then $\hat{k}\cdot\hat{n}\,dA = -\,dx\wedge dy$

Hence

$$\hat{k}\cdot\hat{n}\,dA = \begin{cases} dx\wedge dy & \text{if } \hat{n}=\hat{k} \\ dy\wedge dx & \text{if } \hat{n}=-\hat{k} \end{cases}$$

(orientation of the "surface" domain)

<u>eg2</u> : $\zeta = \zeta_1\,dy\wedge dz + \zeta_2\,dz\wedge dx + \zeta_3\,dx\wedge dy$

Then $d\zeta = d\zeta_1\wedge dy\wedge dz + d\zeta_2\wedge dz\wedge dx + d\zeta_3\wedge dx\wedge dy$

$$= \left(\frac{\partial\zeta_1}{\partial x}dx + \cdots\right)\wedge dy\wedge dz$$

$$+ \left(\cdots + \frac{\partial\zeta_2}{\partial y}dy + \cdots\right)\wedge dz\wedge dx$$

$$+ \left(\cdots + \frac{\partial\zeta_3}{\partial z}dz\right)\wedge dx\wedge dy$$

$$= \left(\frac{\partial\zeta_1}{\partial x} + \frac{\partial\zeta_2}{\partial y} + \frac{\partial\zeta_3}{\partial z}\right)dx\wedge dy\wedge dz$$

$$= \operatorname{div}\vec{F}\,dx\wedge dy\wedge dz$$

where $\vec{F} = \zeta_1\hat{i} + \zeta_2\hat{j} + \zeta_3\hat{k}$.

Hence the divergence thm can be written as

$$\iiint_D d\xi = \iiint_D \left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y} + \frac{\partial \xi_3}{\partial z}\right) \underbrace{dx \wedge dy \wedge dz}_{\text{the oriented volume element}}$$

$$= \iiint_D \operatorname{div} \vec{F} \, dV = \iint_{\partial D} \vec{F} \cdot \underset{\text{outward}}{\hat{n}} \, d\sigma$$

(to be cont'd)