Proof of Divergence Thm

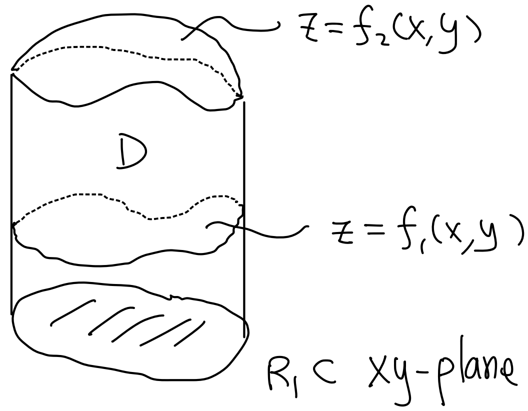
Same as Green's Thm, we'll prove only the case of <u>special domain</u> D which is of type I, II, & III:

$$D = \{(x,y,z) \in \mathbb{R}^3 : (x,y) \in R_1,\ f_1(x,y) \le z \le f_2(x,y)\} \quad (\text{type I})$$

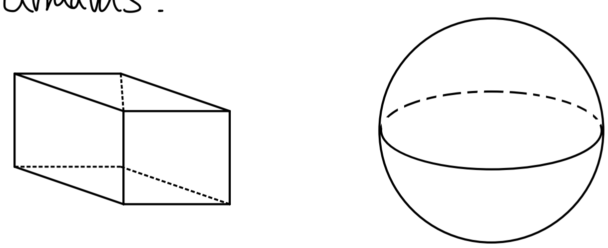
$$= \{(x,y,z) \in \mathbb{R}^3 : (y,z) \in R_2,\ g_1(y,z) \le x \le g_2(y,z)\} \quad (\text{type II})$$

$$= \{(x,y,z) \in \mathbb{R}^3 : (x,z) \in R_3,\ h_1(x,z) \le y \le h_2(x,z)\} \quad (\text{type III})$$

eg. type I domain



eg: special domains:



And also as in the proof of Green's Thm for

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$$

we'll prove 3 equalities in the following which combine to give the Divergence Thm:

$$\begin{cases} \displaystyle\iint_S M\hat{i}\cdot\hat{n}\, d\sigma = \iiint_D \frac{\partial M}{\partial x}\, dV & \text{(by type II)} \\ \displaystyle\iint_S N\hat{j}\cdot\hat{n}\, d\sigma = \iiint_D \frac{\partial N}{\partial y}\, dV & \text{(by type III)} \\ \displaystyle\iint_S L\hat{k}\cdot\hat{n}\, d\sigma = \iiint_D \frac{\partial L}{\partial z}\, dV & \text{(by type I)} \end{cases}$$

The proofs are similar, we'll prove only the last one

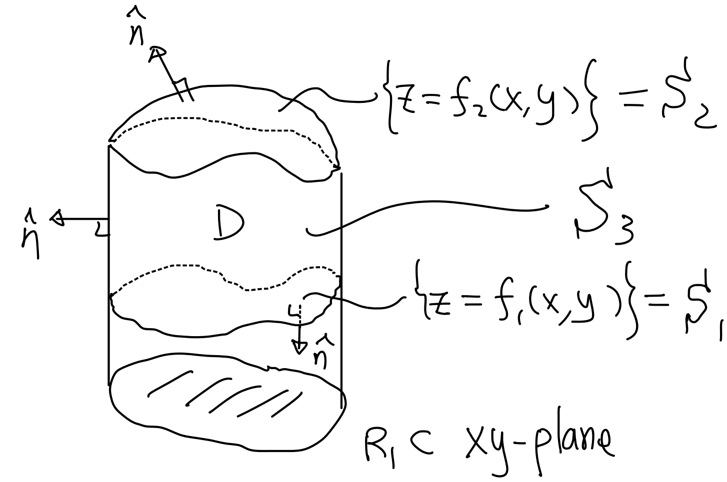
$$\iint_S L\hat{k}\cdot\hat{n}\, d\sigma = \iiint_D \frac{\partial L}{\partial z}\, dV$$

By Fubini's Thm

$$\text{RHS} = \iiint_D \frac{\partial L}{\partial z}\, dV = \iint_{R_1} \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial L}{\partial z}\, dz\right] dxdy \quad \text{(because Type I)}$$

$$= \iint_{R_1} \left[L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))\right] dxdy$$

For the LHS, we note that by definition of type I domain, the boundary surface $S (=\partial D)$ of D can be written as



$$S = S_1 \cup S_2 \cup S_3$$

where $S_1 = \text{graph of } f_1 = \{(x, y, f_1(x,y))\} = \{z = f_1(x,y)\}$

$S_2 = \text{graph of } f_2 = \{(x, y, f_2(x,y))\} = \{z = f_2(x,y)\}$

$S_3 =$ vertical surface (which could be empty) between S_1 & S_2

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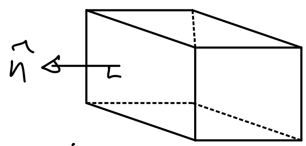
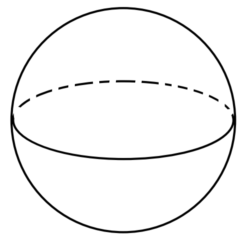


$S_3 =$ union of 4 vertical rectangles



$S_3 = \emptyset$

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Hence $\text{LHS} = \iint_S L\hat{k}\cdot\hat{n}\, d\sigma = \iint_{S_1} L\hat{k}\cdot\hat{n}\, d\sigma + \iint_{S_2} L\hat{k}\cdot\hat{n}\, d\sigma$

$+ \iint_{S_3} L\cancel{\hat{k}\cdot\hat{n}}\, d\sigma$

(Since $\hat{n}$ of a vertical surface is horizontal, hence $\hat{k}\cdot\hat{n} = 0$)

Now on the upper surface $S_2 = \{z = f_2(x,y)\}$

the <u>outward normal</u> $\hat{n}$ is <u>upward</u> (ie $\hat{n}\cdot\hat{k} \geq 0$)

Note that the parametrization

$$(x,y) \longmapsto \vec{r}(x,y) = (x, y, f_2(x,y))$$

has $\begin{cases} \vec{r}_x = \hat{i} + \dfrac{\partial f_2}{\partial x}\hat{k} \\ \vec{r}_y = \hat{j} + \dfrac{\partial f_2}{\partial y}\hat{k} \end{cases}$

and $\vec{r}_x \times \vec{r}_y = -\dfrac{\partial f_2}{\partial x}\hat{i} - \dfrac{\partial f_2}{\partial y}\hat{j} + \hat{k}$ ⟵ $\vec{r}_x \times \vec{r}_y$ is upward

$$\Rightarrow \quad \hat{n} = \frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|} \quad \text{is the upward unit normal}$$

$$\text{and} \quad \hat{k}\cdot\hat{n} = \frac{1}{\|\vec{r}_x \times \vec{r}_y\|}.$$

$$\text{Therefore} \quad \iint_{S_2} L\,\hat{k}\cdot\hat{n}\,d\sigma = \iint_{R_1} L(x,y,f_2(x,y))\,\frac{1}{\cancel{\|\vec{r}_x \times \vec{r}_y\|}}\,\cancel{\|\vec{r}_x \times \vec{r}_y\|}\,dx\,dy$$

$$= \iint_{R_1} L(x,y,f_2(x,y))\,dx\,dy$$

Similarly, note that the <u>outward</u> normal on S_1 (lower surface) is <u>downward</u> (i.e. $\hat{n}\cdot\hat{k} < 0$), we have

$$\hat{n} = -\frac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}, \quad \text{where } \vec{r}(x,y) = (x, y, f_1(x,y))$$
$$(= x\hat{i} + y\hat{j} + f_1(x,y)\hat{k})$$

$$\Rightarrow \quad \hat{k}\cdot\hat{n} = -\frac{1}{\|\vec{r}_x \times \vec{r}_y\|}$$

$$\text{Hence} \quad \iint_{S_1} L\,\hat{k}\cdot\hat{n}\,d\sigma = -\iint_{R_1} L(x,y,f_1(x,y))\,dx\,dy$$

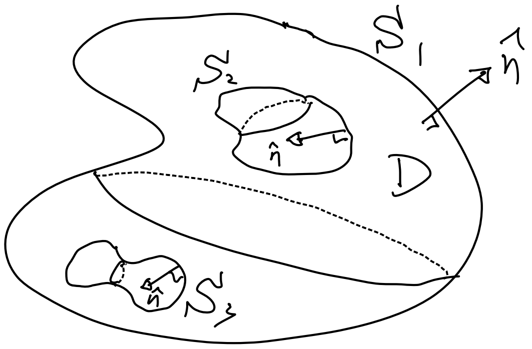
$$\therefore \quad \iint_{S} L\,\hat{k}\cdot\hat{n}\,d\sigma = \iint_{R_1} L(x,y,f_2(x,y))\,dx\,dy - \iint_{R_1} L(x,y,f_1(x,y))\,dx\,dy$$

$$= \iint_{R_1} [L(x,y,f_2(x,y)) - L(x,y,f_1(x,y))]\,dx\,dy$$

$$= \iiint_D \frac{\partial L}{\partial z}\,dV.$$

This completes the proof of the divergence thm. #

<u>Note</u>: Similar to Green's Thm, the Divergence Thm also holds for solid region with finitely many holes insides:

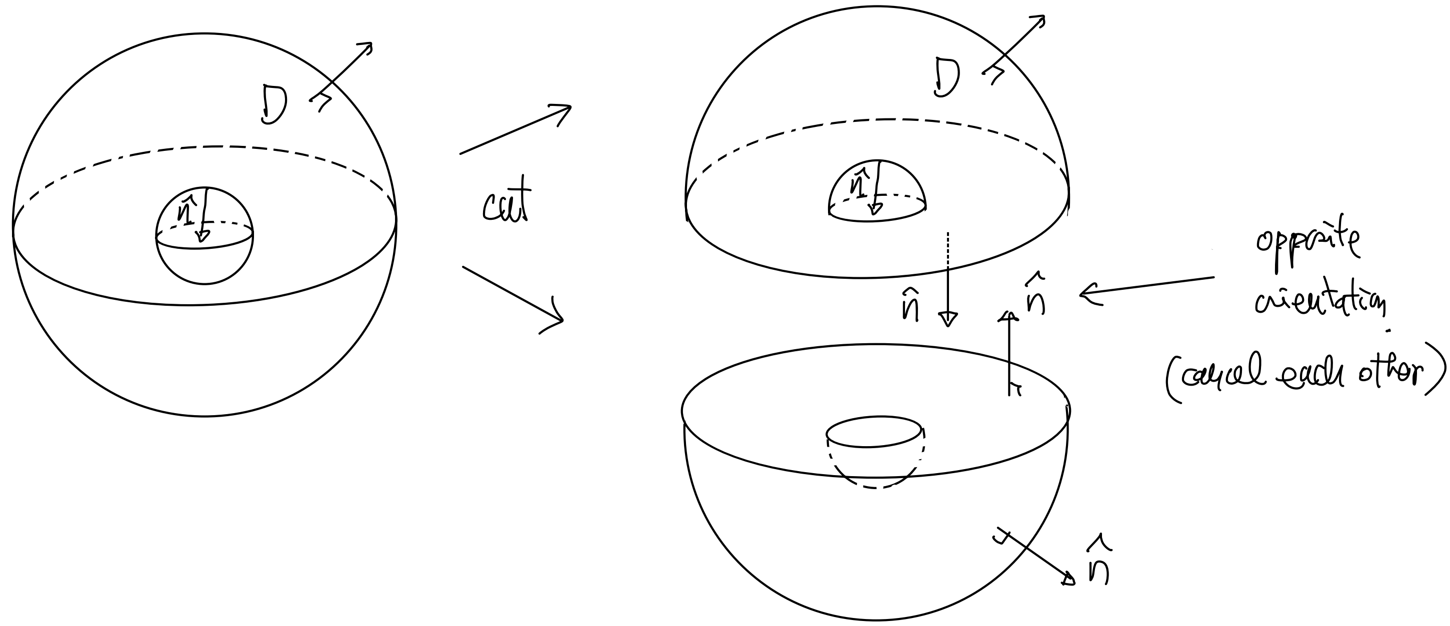


D = solid region inside S_1 but outside of S_2 and S_3

$$\boxed{\iiint_D \vec{\nabla}\cdot\vec{F}\,dV = \sum_{i=1}^{n} \iint_{S_i} \vec{F}\cdot\hat{n}\,d\sigma}$$

for $\hat{n}$ = <u>outward</u> normal <u>with respect to D</u>.

eg (idea of proof of this kind of surface :)



Note: Physical meaning of $\operatorname{div} \vec{F} = \vec{\nabla} \cdot \vec{F}$ in $\mathbb{R}^3$

$=$ flux density (by the divergence thm)

Unified treatment of Green's, Stokes', and Divergence Theorems

Stokes' Thm in notations of differential forms (in $\mathbb{R}^3$)

Working definition of differential forms

(1) A differential 1-form (or simply 1-form)

is a linear combination of the symbols dx, dy & dz:

$$\boxed{\omega = \omega_1\,dx + \omega_2\,dy + \omega_3\,dz}$$

with coefficients $\omega_1, \omega_2, \omega_3$ functions on $\mathbb{R}^3$.

eg: Total different of a function f on $\mathbb{R}^3$:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz \quad \text{is a 1-form.}$$

(2) Wedge product: Let "$\wedge$" be an operation such that

$$\begin{cases} dx\wedge dx = dy\wedge dy = dz\wedge dz = 0 \\ dx\wedge dy = -\,dy\wedge dx \\ dy\wedge dz = -\,dz\wedge dy \\ dz\wedge dx = -\,dx\wedge dz \end{cases}$$

and satisfies other usual rules in arithmetic.

i.e. If $\omega = \omega_1\,dx + \omega_2\,dy + \omega_3\,dz$

$\eta = \eta_1\,dx + \eta_2\,dy + \eta_3\,dz$

then we have

$$\omega \wedge \eta = (\omega_1 dx + \omega_2 dy + \omega_3 dz) \wedge (\eta_1 dx + \eta_2 dy + \eta_3 dz)$$

$$= (\omega_1 dx)\wedge(\eta_1 dx) + (\omega_2 dy)\wedge(\eta_1 dx) + (\omega_3 dz)\wedge(\eta_1 dx)$$
$$+ (\omega_1 dx)\wedge(\eta_2 dy) + (\omega_2 dy)\wedge(\eta_2 dy) + (\omega_3 dz)\wedge(\eta_2 dy)$$
$$+ (\omega_1 dx)\wedge(\eta_3 dz) + (\omega_2 dy)\wedge(\eta_3 dz) + (\omega_3 dz)\wedge(\eta_3 dz)$$

$$= (\omega_1\eta_1)\cancel{dx\wedge dx} + \omega_2\eta_1\, dy\wedge dx + \omega_3\eta_1\, dz\wedge dx$$
$$+ (\omega_1\eta_2)\, dx\wedge dy + \omega_2\eta_2\,\cancel{dy\wedge dy} + \omega_3\eta_2\, dz\wedge dy$$
$$(\omega_1\eta_3)\, dx\wedge dz + \omega_2\eta_3\, dy\wedge dz + \omega_3\eta_3\,\cancel{dz\wedge dz}$$

$$\therefore \boxed{\begin{aligned}\omega\wedge\eta = &(\omega_2\eta_3 - \omega_3\eta_2)\, dy\wedge dz \\ &+ (\omega_3\eta_1 - \omega_1\eta_3)\, dz\wedge dx \\ &+ (\omega_1\eta_2 - \omega_2\eta_1)\, dx\wedge dy\end{aligned}}$$

- Linear combinations of $dy\wedge dz$, $dz\wedge dx$ & $dx\wedge dy$ are called <u>differential 2-forms</u> (on $\mathbb{R}^3$)

$$\boxed{\zeta = \zeta_1\, dy\wedge dz + \zeta_2\, dz\wedge dx + \zeta_3\, dx\wedge dy}$$

Similarly, if ω is a 1-form and

ζ is a 2-form

then we can define $\omega\wedge\zeta$

eg: If $\omega = dx$, $\zeta = dy \wedge dz$

then $\omega \wedge \zeta = dx \wedge dy \wedge dz$

Note that we insist on the anti-commutativity of wedge product,

we have

$$\begin{aligned} dx \wedge dy \wedge dz &= -\, dy \wedge dx \wedge dz \\ &= \, dy \wedge dz \wedge dx \\ &= -\, dz \wedge dy \wedge dx \\ &= \, dz \wedge dx \wedge dy \\ &= -\, dx \wedge dz \wedge dy \end{aligned}$$

And $dx \wedge dx \wedge dy = \cdots = 0$ whenever one of the dx, dy, dz is repeated.

Hence, as $\dim \mathbb{R}^3 = 3$, all "linear combinations" of "3-forms" are just $f\, dx \wedge dy \wedge dz$

which is called a differential 3-form (also called a volume form if $f > 0$)

Note: It is convenient to call smooth functions f the differential 0-form.

Summary (on $\mathbb{R}^3$)

$$0\text{-form} = f$$

$$1\text{-form} = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz$$

$$2\text{-form} = \xi_1\, dy \wedge dz + \xi_2\, dz \wedge dx + \xi_3\, dx \wedge dy$$

$$3\text{-form} = g\, dx \wedge dy \wedge dz$$

where, f, g, ω_i, ξ_i are (smooth) functions

Note: One can certainly define k-form for any $k \geq 0$. But in $\mathbb{R}^3$, k-forms are zero for $k > 3$:

$$dx^i \wedge dx \wedge dy \wedge dz = 0, \quad \text{where } dx^i = dx, dy, \text{ or } dz.$$