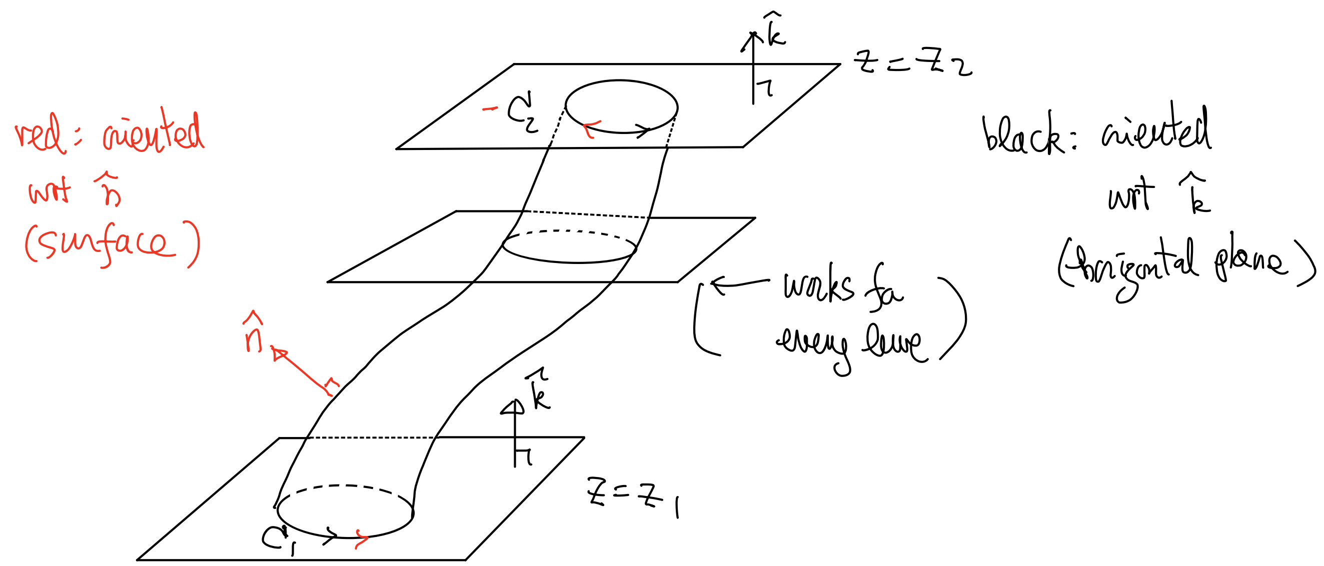
eg 63: Let $\vec{F}$ be a vector field such that $\underline{\vec{\nabla}\times\vec{F}=\vec{0}}$

and defined on a region containing the surface S

with unit normal vector field $\hat{n}$ as in the figure:



The boundary C of S has 2 components C_1 & C_2 at the level

$z=z_1$ & $z=z_2$ respectively.

If both C_1, C_2 oriented anti-clockwisely with respect to the

"horizontal planes" (i.e. $\hat{k}$)

Then when C oriented anti-clockwisely with respect to the

surface normal $\hat{n}$, we have

$$C = C_1 - C_2$$

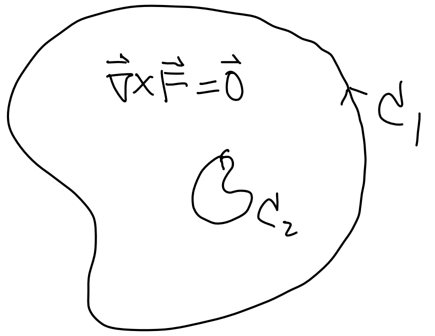
And Stokes' Thm $\Rightarrow$

$$0=\iint_S(\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma=\oint_C\vec{F}\cdot d\vec{r}=\oint_{C_1}\vec{F}\cdot d\vec{r}-\oint_{C_2}\vec{F}\cdot d\vec{r}$$

$$\Rightarrow\quad \oint_{C_1}\vec{F}\cdot d\vec{r}=\oint_{C_2}\vec{F}\cdot d\vec{r}$$

$\#$

Compare this with Green's Thm on plane region with one hole :



$$\Rightarrow \quad \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_2} \vec{F}\cdot d\vec{r} \qquad \text{(check!)}$$

anti-clockwisely wrt "plane" (not see region as a surface).

Proof of Thm 10 (3-dim'l case)

Only the "$\Leftarrow$" part remains to be proved:

By assumption $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ satisfies the system of eqts in the cor. to the Thm 9, that is

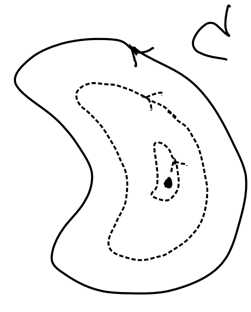
$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}, \quad \frac{\partial N}{\partial z} = \frac{\partial L}{\partial y}, \text{ and } \frac{\partial L}{\partial x} = \frac{\partial M}{\partial z}.$$

Hence $\vec{\nabla}\times\vec{F} = \vec{0}$

Let C be a simple closed curve in a simply-connected region D

Then C can be deformed to a point inside D

The process of deformation gives an oriented surface $S \subset D$ such that the boundary of $S = C$.

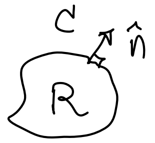
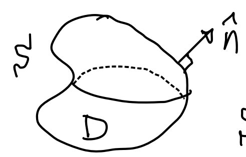


By Stokes' Thm,

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot \hat{n}\, d\sigma = 0 \quad (\text{since } \vec{\nabla}\times\vec{F} = \vec{0})$$

Then Thm 9 $\Rightarrow$ $\vec{F}$ is conservative. ##

Summary

$n=2$	$n=3$
<u>Tangential form</u> of Green's Thm	<u>Stokes' Thm</u>
$\oint_C \vec{F}\cdot d\vec{r} = \iint_R (\vec{\nabla}\times\vec{F})\cdot \hat{k}\, dA$	$\oint_C \vec{F}\cdot d\vec{r} = \iint_S (\vec{\nabla}\times\vec{F})\cdot \hat{n}\, d\sigma$
<u>Normal form</u> of Green's Thm	<u>Divergence Thm</u> (next topic)
$\oint_C \vec{F}\cdot \hat{n}\, ds = \iint_R \vec{\nabla}\cdot\vec{F}\, dA$	$\iint_S \vec{F}\cdot \hat{n}\, d\sigma = \iiint_D \vec{\nabla}\cdot\vec{F}\, dV$
 $C = \partial R$	 $S = \partial D$
"flux": by definition, $\hat{n}$ is the "outward" unit normal of the curve "C" in the "plane".	"flux": $\hat{n}$ is the "outward" unit normal to the surface S encloses a solid region D

Thm 13 (Divergence Theorem)

Let $\vec{F}$ be a C^1 vector field on $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^3$ (no boundary)

S be a piecewise smooth oriented closed surface

enclosing a (solid) region $D \subseteq \Omega$.

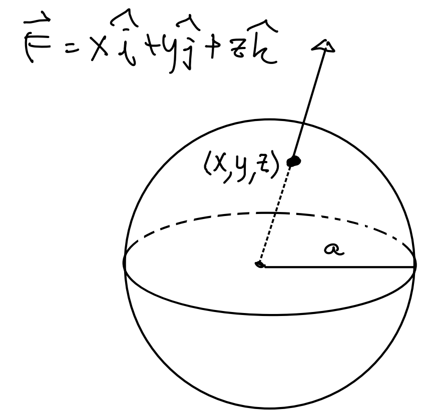
Let $\hat{n}$ be the outward pointing unit normal vector field on S,

Then
$$\boxed{\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iiint_D \operatorname{div}\vec{F}\,dV = \iiint_D \vec{\nabla}\cdot\vec{F}\,dV}$$

eg 64 Verify Divergence Thm for

$$\vec{F} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$S = \{x^2+y^2+z^2 = a^2\} \quad (a>0)$$



(surface $= S_a^2$ 2-dim'l sphere of radius a centered at $(0,0,0)$)

$D =$ solid ball bounded by S.

Soln: At $(x,y,z) \in S$

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k})$$ is the outward pointing unit normal

$$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_S (x\hat{i}+y\hat{j}+z\hat{k})\cdot\frac{1}{a}(x\hat{i}+y\hat{j}+z\hat{k})\,d\sigma = a\iint_S d\sigma$$

$= 4\pi a^3$ (Check!)

On the other hand,

$$\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(x\hat{i}+y\hat{j}+z\hat{k}\right)$$

$$= \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3$$

$$\Rightarrow \iiint_D \operatorname{div}\vec{F}\,dV = 3\iiint_D dV = 3\cdot\frac{4\pi a^3}{3} = 4\pi a^3$$

$$= \iint_S \vec{F}\cdot\hat{n}\,d\sigma$$

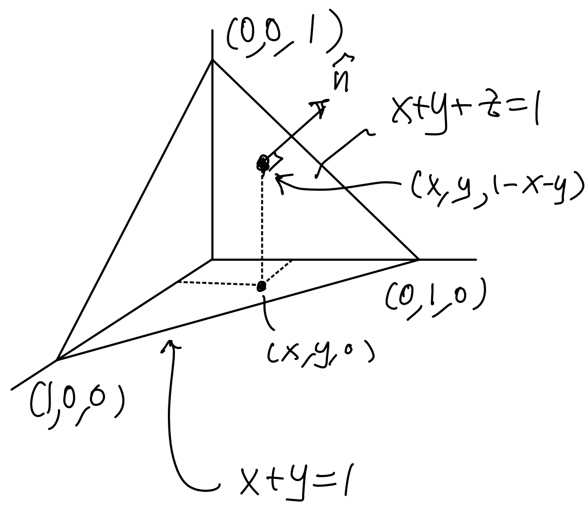
#

eg65 $\vec{F} = x\sin y\,\hat{i} + (\cos y + z)\hat{j} + z^2\hat{k}$

Compute outward flux of $\vec{F}$ across the boundary ∂T of

$$T = \left\{(x,y,z)\in\mathbb{R}^3 : \begin{matrix} x+y+z\le 1 \\ x,y,z\ge 0\end{matrix}\right\}$$

(tetrahedron)



Soln: $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x\sin y) + \frac{\partial}{\partial y}(\cos y + z) + \frac{\partial}{\partial z}(z^2)$

$= 2z$ (Check!)

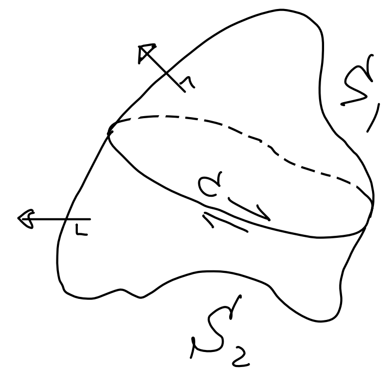
Divergence Thm $\Rightarrow$

$$\iint_{\partial T}\vec{F}\cdot\hat{n}\,d\sigma = \iiint_T \operatorname{div}\vec{F}\,dV = \int_0^1\int_0^{1-x}\int_0^{1-x-y} 2z\,dz\,dy\,dx = \frac{1}{12} \quad \text{(Check!)}$$

#

eg66: Let S_1, S_2 be 2 surfaces with common boundary curve C such that $S_1 \cup S_2$ forms a closed surface enclosing a solid region D (without hole)

Suppose $\hat{n}$ is the outward unit normal of the (boundary of) solid region D.



Then the orientation of C with respect to $(S_1, \hat{n})$ and $(S_2, \hat{n})$ are opposite (since "$\hat{b}$" of S_1 & S_2 are opposite)

Find

$$\iiint_D \operatorname{div}(\vec{\nabla}\times\vec{F})\, dV,$$

where $\vec{F}$ is a C^2 vector field on D.

Soln

$$\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \oint_C \vec{F}\cdot d\vec{r} \longleftarrow \text{(+ve oriented wrt } (S_1, \hat{n}))$$

$$= -\oint_C \vec{F}\cdot d\vec{r} \longleftarrow \text{(+ve oriented wrt } (S_2, \hat{n}))$$

$$= -\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma$$

$$\Rightarrow \iint_{S_1\cup S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = 0 \quad \text{(see eg 61(c) for explicit example)}$$

Divergence Thm $\Rightarrow$ $$\iiint_D \operatorname{div}(\vec{\nabla}\times\vec{F})\, dV = \iint_{S_1\cup S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = 0$$

<u>Remark</u> The result holds for any C^2 vector field $\vec{F}$ defined on any D. It is <u>consistent</u> with

(Ex!) $$\boxed{\vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0}$$ $\forall$ C^2 vector field

i.e. $$\boxed{\operatorname{div}(\operatorname{curl} \vec{F}) = 0}$$

<u>Compare</u> :

$$\boxed{\operatorname{curl}(\operatorname{grad} f) = \vec{0} \quad \text{i.e.} \quad \vec{\nabla} \times (\vec{\nabla} f) = \vec{0}}$$