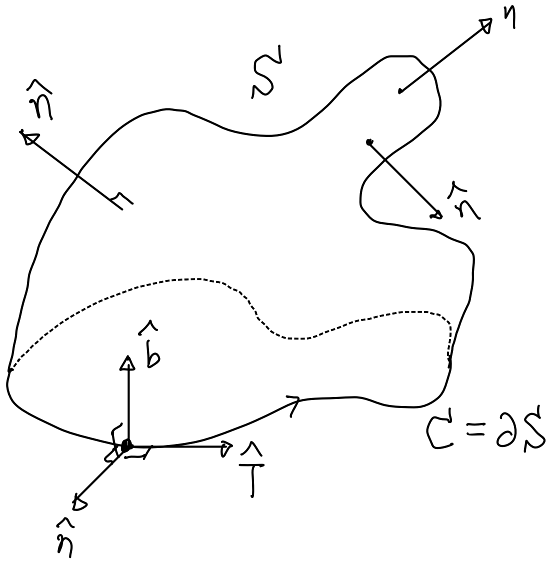
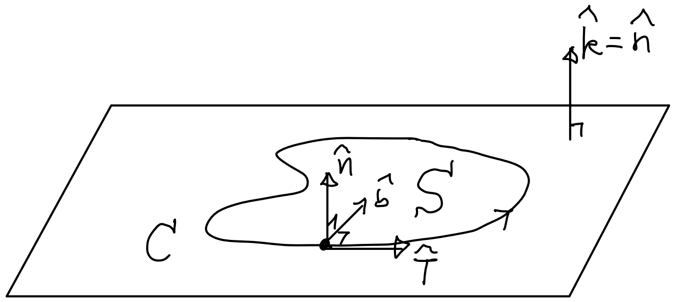
eg 60

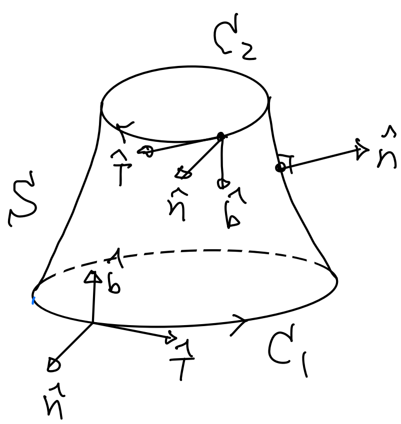
(1)



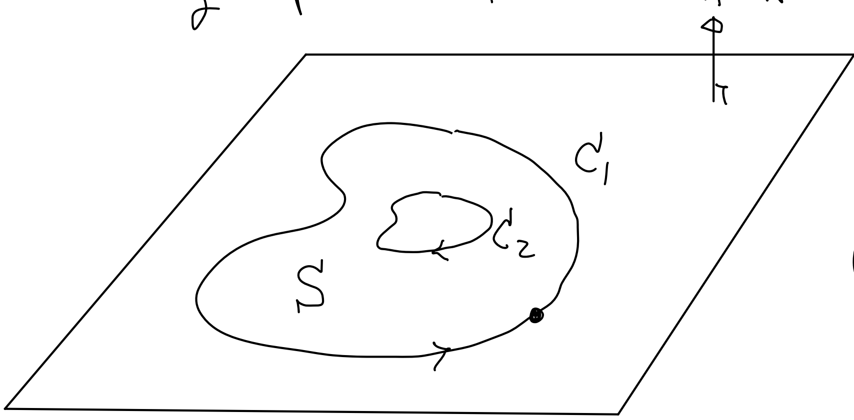
(2) $S \subset \mathbb{R}^2$ with $\hat{n} = \hat{k}$:
same as the usual anti-clockwise direction of a closed plane curve.



(3) 2 boundary components



(4) 2 boundary components $S \subset \mathbb{R}^2$ $\hat{n} = \hat{k}$



(Ex!)

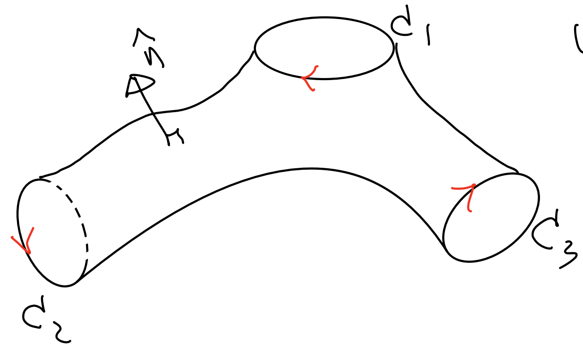
Important remark: If S is a region in $\mathbb{R}^2$, then a boundary component of S (C_1 or C_2 for instance) has "2" concepts of "oriented anti-clockwisely" with respect to S = region and $\mathbb{R}^2$.

Even S and $\mathbb{R}^2$ have the same orientation, i.e. $\hat{n} = \hat{k}$, we still have the following situations: (C_1, C_2 as in figure above)

	S (region)	$\mathbb{R}^2$
C_1	anti-clockwise (+)	anti-clockwise (+)
C_2	anti-clockwise (+)	clockwise (−)

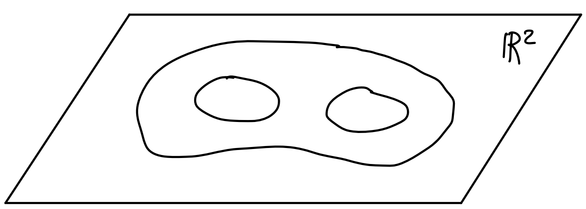
(5)

(Ex!)



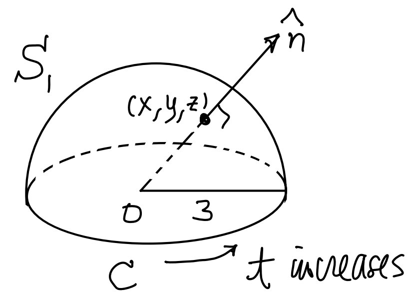
what is the oriented of C_i s.t. their oriented anti-clockwise with respect to $\hat{n}$? (Ex!)

(Similarly optional:



?)

eg 61 Verifying Stokes' Thm



(a) $S_1 : x^2+y^2+z^2=9,\ z\geq 0$

with upward normal $\hat{n}$ (i.e. $\hat{k}$-component >0)

boundary $C : x^2+y^2=9,\ z=0$

And $\vec{F}=y\hat{i}-x\hat{j}$

Parametrize C by $\vec{r}(t)=(3\cos t, 3\sin t, 0)$, $0\leq t\leq 2\pi$

$$= 3\cos t\,\hat{i}+3\sin t\,\hat{j}$$

(has the correct direction, i.e. oriented anti-clockwisely wrt $\hat{n}$)

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi}(3\sin t\,\hat{i}-3\cos t\,\hat{j})\cdot(-3\sin t\,\hat{i}+3\cos t\,\hat{j})\,dt$$

$$= -18\pi \quad \text{(check!)}$$

For the surface integral :

$$\vec{\nabla}\times\vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & -x & 0 \end{vmatrix} = -2\hat{k} \quad \text{(Check!)}$$

Since S_1 is a hemisphere (upper) centered at origin of radius 3

$$\hat{n} = \frac{x\hat{i}+y\hat{j}+z\hat{k}}{\sqrt{x^2+y^2+z^2}} = \frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k}) \quad \left(\begin{matrix} z\geq 0 \\ \Updownarrow \\ \text{upward}\end{matrix}\right)$$

The surface S_1 can be regarded as a level surface given by

$$g(x,y,z) = x^2+y^2+z^2 = 9$$

$$\Rightarrow \quad \vec{\nabla} g = (2x, 2y, 2z)$$

Since $z>0$ (except the boundary) on S_1, $\dfrac{\partial g}{\partial z} = 2z \neq 0$

Hence $\quad d\sigma = \dfrac{|\vec{\nabla} g|}{\left|\frac{\partial g}{\partial z}\right|}\,dx\,dy = \dfrac{\sqrt{4x^2+4y^2+4z^2}}{|2z|}\,dx\,dy = \dfrac{3}{z}\,dx\,dy$

Therefore $\displaystyle\iint_{S_1} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$

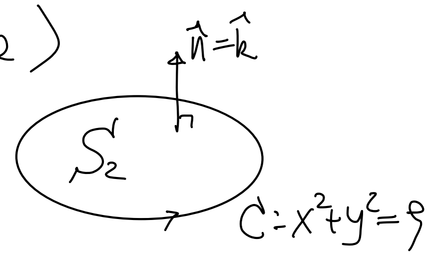
$$= \iint_{\{x^2+y^2\le 9\}} (-2\hat{k})\cdot\frac{1}{3}(x\hat{i}+y\hat{j}+z\hat{k})\,\frac{3}{z}\,dx\,dy$$

$$= \iint_{\{x^2+y^2\le 9\}} (-2)\,dx\,dy = -18\pi \qquad \text{(check!)}$$

※

(b) (Same C & same $\vec{F}$, but new surface)

$S_2 : x^2+y^2<9, \quad z=0$

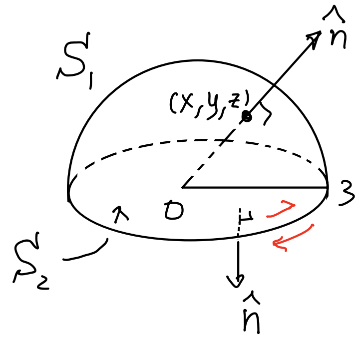


$$\iint_{S_2} (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma = \iint_{\{x^2+y^2<9\}} (-2\hat{k})\cdot\hat{k}\,dx\,dy$$

$$= -2\,\text{Area}(\{x^2+y^2<9\}) = -18\pi.$$

(c) Same $\vec{F} = y\hat{i} - x\hat{j}$

$S_3 = S_1 \cup S_2$



S_3 has no boundary and in fact encloses a solid region.

Suppose $\hat{n}$ = outward normal of the solid.

$$\iint_{S_3} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma = \underbrace{\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma}_{\text{(same as eg (a))}} + \underbrace{\iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma}_{\text{(opposite to eg (b))}}$$

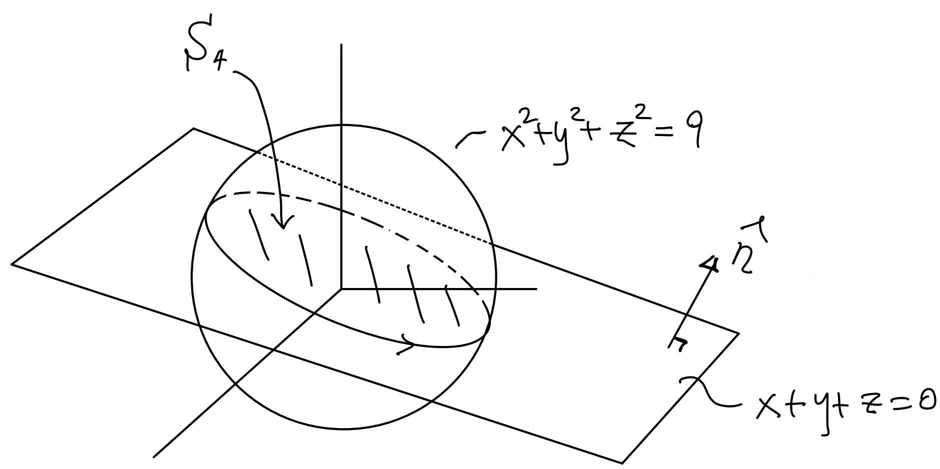
$$= \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma + \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot (-\hat{k}) \, d\sigma$$

$$= -18\pi - (-18\pi) = 0$$

$$\left(= \oint_C \vec{F} \cdot d\vec{r} - \oint_C \vec{F} \cdot d\vec{r} = 0 \right)$$

\#

<u>eg62</u> Let $\vec{F} = y\hat{i} - x\hat{j}$ (same $\vec{F}$ as in eg61, new surface & new boundary curve)



$S_4' = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 \leq 9,\ x+y+z=0\}$

boundary curve of S_4' : $C_4 = \{(x,y,z) \in \mathbb{R}^3 : x^2+y^2+z^2 = 9,\ x+y+z=0\}$

Find $\oint_{C_4'} \vec{F} \cdot d\vec{r}$ (with direction of C_4' given as in the figure.)

Soln: Apply Stokes' Thm

$$\oint_{C_4} \vec{F} \cdot d\vec{r} = \iint_{S_4} (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

$\hat{n}$ is the normal to S_4' is given by the coefficients of the equation of the plane

i.e. $\hat{n} = \dfrac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

+ve ⇓ upward

$$= \iint_{S_4'} (-2\hat{k}) \cdot \frac{\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}\, d\sigma$$

$$= -\frac{2}{\sqrt{3}} \iint_{S_4} d\sigma = -\frac{2}{\sqrt{3}} \mathrm{Area}(S_4')$$

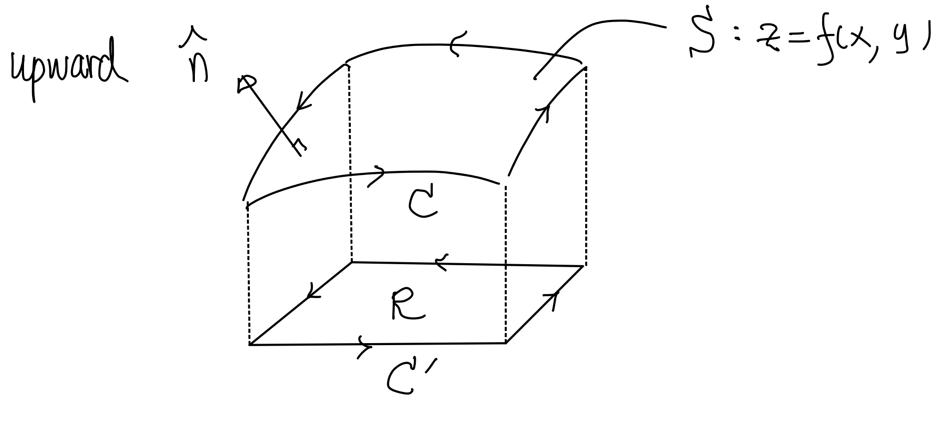
$$= -\frac{2}{\sqrt{3}} (\pi \cdot 3^2) = \frac{-18\pi}{\sqrt{3}}$$

⋕

Proof of Stokes' Thm

Special case: S is a graph given by

$z = f(x,y)$ over a region R with upward normal



Assume C is the boundary of S, and C' is the boundary of R (anti-clockwisely oriented wrt the normal of S and the plane respectively)

Parametrize the graph as

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}, \qquad (x,y) \in R$$

Then as before $\begin{cases} \vec{r}_x = \hat{i} + f_x \hat{k} \\ \vec{r}_y = \hat{j} + f_y \hat{k} \end{cases}$

$$\vec{r}_x \times \vec{r}_y = -f_x \hat{i} - f_y \hat{j} + \hat{k}$$ +ve $\Rightarrow$ upward.

Hence $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_y}{\|\vec{r}_x \times \vec{r}_y\|}$ is the upward (unit) normal of S,

and $d\sigma = \|\vec{r}_x \times \vec{r}_y\| \, dx\,dy = \|\vec{r}_x \times \vec{r}_y\| \, dA$ area element of R.

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ the C^1 vector field.

Then $$\iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\, d\sigma = \iint_R (\vec{\nabla}\times\vec{F})(\vec{r}(x,y)) \cdot \frac{\vec{r}_x\times\vec{r}_y}{\cancel{\|\vec{r}_x\times\vec{r}_y\|}} \cancel{\|\vec{r}_x\times\vec{r}_y\|}\, dA$$

$$= \iint_R \left[(L_y - N_z)\hat{i} + (M_z - L_x)\hat{j} + (N_x - M_y)\hat{k}\right]\cdot\left(-f_x\hat{i} - f_y\hat{j} + \hat{k}\right) dA$$

$$= \iint_R \left[-f_x(L_y - N_z) - f_y(M_z - L_x) + (N_x - M_y)\right] dA$$

For the line integral

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C M\,dx + N\,dy + L\,dz$$

$$= \oint_{C'} M\,dx + N\,dy + L\,df \qquad (z = f(x,y))$$

$$= \oint_{C'} M\,dx + N\,dy + L(f_x\,dx + f_y\,dy)$$

$$= \oint_{C'} (M + f_x L)\,dx + (N + f_y L)\,dy$$

Precisely: If C' is parametrized by $\vec{\gamma}(t) = (x(t), y(t))$ for $a \le t \le b$

Then C is parametrized by

$$\vec{r}(t) = (x(t), y(t), f(x(t), y(t))) \qquad a \le t \le b$$

$$\Rightarrow \oint_C \vec{F}\cdot d\vec{r} = \int_a^b \left[M(\vec{r}(t))\,x'(t) + N(\vec{r}(t))\,y'(t) + L(\vec{r}(t))\,\frac{d}{dt} f(x(t), y(t)) \right] dt$$

$$= \int_a^b \left[M x' + N y' + L (f_x x' + f_y y') \right] dt$$

$$= \int_a^b \left[(M + f_x L) x' + (N + f_y L) y' \right] dt$$

$$= \oint_{C'} (M + f_x L)\, dx + (N + f_y L)\, dy$$

Then by Green's Thm

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_{C'} (M + f_x L)\, dx + (N + f_y L)\, dy$$

$$= \iint_R \left[\frac{\partial}{\partial x}(N + f_y L) - \frac{\partial}{\partial y}(M + f_x L) \right] dA$$

$$= \iint_R \left[\begin{array}{l} \frac{\partial}{\partial x}\left(N(x,y,f(x,y)) + f_y(x,y) L(x,y,f(x,y)) \right) \\ - \frac{\partial}{\partial y}\left(M(x,y,f(x,y)) + f_x(x,y) L(x,y,f(x,y)) \right) \end{array} \right] dA$$

$$= \iint_R \left[\begin{array}{l} (N_x + N_z f_x) + \left(\cancel{f_{yx}} L + f_y (L_x + \cancel{L_z f_x}) \right) \\ - (M_y + M_z f_y) - \left(\cancel{f_{xy}} L + f_x (L_y + \cancel{L_z f_y}) \right) \end{array} \right] dA$$

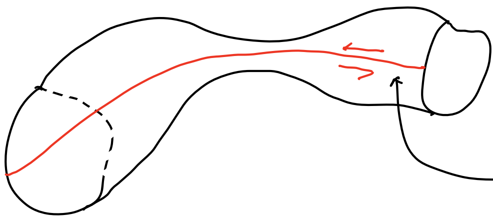
$$= \iint_R \left[-f_x (L_y - N_z) - f_y (M_z - L_x) + (N_x - M_y) \right] dA$$

$$= \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n}\, d\sigma$$

This proves the case of C^2 graph.

General case: Divides S into finitely many pieces which are graphs (in certain projection).

[This includes S with many boundary components as in the Green's Thm



add some curve likes this to make it in 1 bdy component.]

(Proof of general case omitted)

Note: Stokes' Thm applies to surfaces like the following