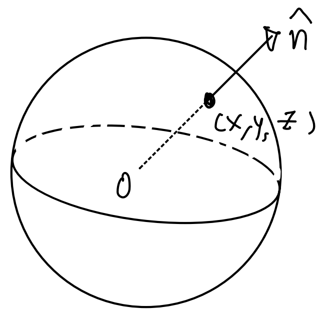
Orientation of Surfaces

To integrate vector fields over surfaces, we need

> **Def 17** (Orientation of a surface in $\mathbb{R}^3$)
>
> A surface S is orientable if one can define a unit normal vector field continuously at every point of S.
>
> (Such a chosen normal vector field is called an orientation of S)

eg 57: (i) $S^2 = \{x^2+y^2+z^2=1\}$



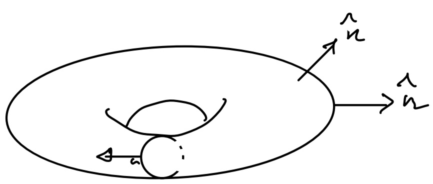
$\hat{n} = x\hat{i}+y\hat{j}+z\hat{k}$ is cts

$(\|\hat{n}\| = \sqrt{x^2+y^2+z^2} = 1)$

$\therefore$ S is orientable

(and the chosen $\hat{n}$ is called an orientation)

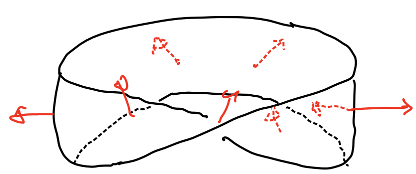
(ii) Torus



orientable

(iii) Möbius strip

is not orientiable



(usually refered as a surface of one side)

<u>Remark</u>: Parametric surface $\underline{S = \vec{r}(u,v)}$ are <u>always orientable</u>:

the unit normal vector field $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ given by

the parametrization is a <u>continuous unit normal</u> vector field on S.

($\vec{r}_u, \vec{r}_v$ "continuous" tangent vectors $\Rightarrow$ $\vec{r}_u \times \vec{r}_v$ is a "continuous" normal vector

$\|\vec{r}_u \times \vec{r}_v\| \neq 0$ $\Rightarrow$ $\dfrac{\vec{r}_u \times \vec{r}_v}{\|\vec{r}_u \times \vec{r}_v\|}$ <u>continuous unit normal</u> vector field)

<u>Terminology</u>

Given a connected <u>orientable</u> surface $S \subset \mathbb{R}^3$, there are

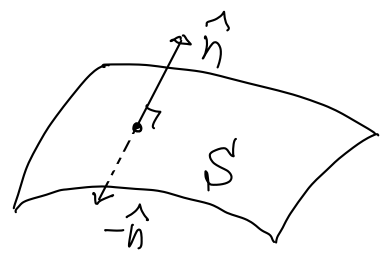
<u>two ways to assign</u> the continuous unit normal vector field

Suppose S is orientable and

we have already <u>chosen one</u> continuous

unit normal vector field $\hat{n}$,

before choosing a parametrization.



<u>Recall</u>: We said that a parametrization $\vec{r}(u,v)$ of S

is <u>compatible</u> with the <u>orientation</u> of S given by

the unit normal vector field $\hat{n} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$.

(The <u>chosen unit normal</u> vector field is also referred as

the <u>orientation</u> of S)

Def 19: Let S be orientable with unit normal $\hat{n}$ (continuous).

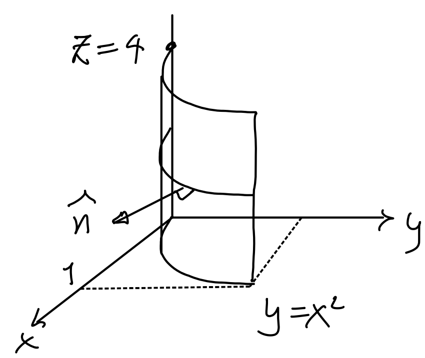
Let $\vec{F}$ be a vector field on S.

Then the flux of $\vec{F}$ across S is

$$\text{Flux} = \iint_S \vec{F} \cdot \hat{n} \, d\sigma$$

eg 59: S:

$$y = x^2, \quad 0 \le x \le 1, \quad 0 \le z \le 4$$



with $\hat{n}$ given by the natural parametrization

$$\vec{r}(x,z) = x\hat{i} + x^2\hat{j} + z\hat{k}$$

Let $\vec{F} = yz\hat{i} + x\hat{j} - z^2\hat{k}$

Find $\iint_S \vec{F} \cdot \hat{n} \, d\sigma$

Soln: To calculate $\hat{n} = \dfrac{\vec{r}_x \times \vec{r}_z}{\|\vec{r}_x \times \vec{r}_z\|}$, we have

$$\begin{cases} \vec{r}_x = \hat{i} + 2x\hat{j} \\ \vec{r}_z = \hat{k} \end{cases} \Rightarrow \vec{r}_x \times \vec{r}_z = 2x\hat{i} - \hat{j}$$

$$\therefore \quad \hat{n} = \frac{2x\hat{i} - \hat{j}}{\sqrt{4x^2+1}}$$

Then $$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \int_0^4\int_0^1 \underbrace{(yz\hat{i}+x\hat{j}-z^2\hat{k})}_{\vec{F}}\cdot \underbrace{\frac{2x\hat{i}-\hat{j}}{\cancel{\sqrt{4x^2+1}}}}_{\hat{n}}\cdot\underbrace{\cancel{\sqrt{4x^2+1}}\,dx\,dz}_{d\sigma}$$

$$= \int_0^4\int_0^1 (2x^3z-x)\,dx\,dz = 2 \quad (\text{Check!})$$

#

<u>Remark</u>: $$\iint_S \vec{F}\cdot\hat{n}\,d\sigma = \iint_{(u,v)} \vec{F}(\vec{r}(u,v))\cdot\frac{\vec{r}_u\times\vec{r}_v}{\|\vec{r}_u\times\vec{r}_v\|}\,\|\vec{r}_u\times\vec{r}_v\|\,du\,dv$$

$$= \iint_{(u,v)} \vec{F}(\vec{r}(u,v))\cdot(\vec{r}_u\times\vec{r}_v)\,du\,dv$$

<u>Thm 12</u> (<u>Stokes' Theorem</u>)

Let S be a piecewise smooth <u>oriented</u> surface with piecewise smooth boundary C (including the case that C is a union of finitely many curves). Let

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \quad \text{be a } C^1 \text{ vector field.}$$

Suppose C is <u>oriented anti-clockwisely with respect to</u> the <u>unit normal vector field</u> $\hat{n}$ on S. Then

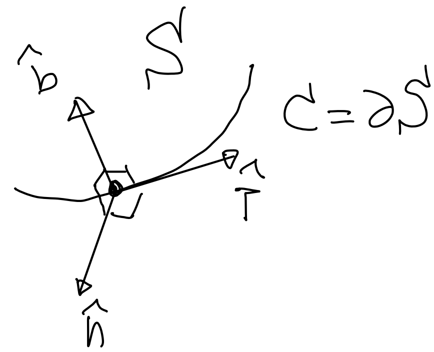
$$\oint_C \vec{F}\cdot d\vec{r} = \iint_S \operatorname{curl}\vec{F}\cdot\hat{n}\,d\sigma = \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$

Here : (i) if $C = C_1 \cup \cdots \cup C_k$, then it means

$$\sum_{i=1}^{k} \oint_{C_i} \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot \hat{n} \, d\sigma$$

(ii) "C is oriented anti-clockwisely with respect to the unit normal vector field $\hat{n}$" means that we choose the <u>direction</u> of C such that its (unit) tangent vector $\hat{T}$ satisfies

$$\boxed{\hat{b} = \hat{n} \times \hat{T} \text{ pointing } \underline{\text{toward}} \text{ the surface } S}$$



i.e. the unit vector $\hat{b}$ tangent to S, normal to C and pointing toward S satisfies

$$\boxed{\hat{T} = \hat{b} \times \hat{n}}$$