Implicit Surface (Level surface)

Suppose S is given by $F(x,y,z) = c$

i.e. $S = F^{-1}(c)$

(Note: F is a function of 3-variables, not vector field)

eg53: $F(x,y,z) = x^2+y^2+z^2$

Is $F^{-1}(0)$ a surface?

Soln: No, since $F^{-1}(0) = \{(0,0,0)\}$ not a surface.

Remark: If $\vec{\nabla} F \neq \vec{0}$ at a point, then IFT implies that

$S = F^{-1}(c)$ is a "surface" (c = value of F at that point)

near that point (in fact, a graph!)

eg53 (cont'd) $\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} + 2z\hat{k}$

$\vec{\nabla} F = \vec{0} \Leftrightarrow (x,y,z) = (0,0,0)$

Hence if $c > 0$, then $\forall (x,y,z) \in F^{-1}(c)$, we have

$$\vec{\nabla} F(x,y,z) \neq \vec{0}$$

$\Rightarrow S = F^{-1}(c)$ is a surface $\forall c > 0$

(What are these surfaces?)

Terminology : $S = F^{-1}(c)$ is said to be smooth if (1) F is C^1 on S, and (2) $\vec{\nabla} F \neq \vec{0}$ on S.

How to compute surface area for a smooth level surface $S = F^{-1}(c)$?

By $\vec{\nabla} F \neq \vec{0}$, at least one of the partial derivatives F_x, F_y, F_z is nonzero. Let assume $F_z = \frac{\partial F}{\partial z} \neq 0$ (the other cases are similar)

IFT $\Rightarrow$ $S = F^{-1}(c) = \{ F(x,y,z) = c \}$

can be written (locally) as a graph

$z = f(x,y)$ (near a point)

i.e. $F(x, y, f(x,y)) = c$ ($\forall (x,y)$ near a point)

Then chain rule $\Rightarrow$ $\begin{cases} f_x = -\dfrac{F_x}{F_z} \\ f_y = -\dfrac{F_y}{F_z} \end{cases}$ $(F_z \neq 0)$

Hence $\text{Area}(S) = \iint_\Omega \sqrt{1 + |\vec{\nabla} f|}\, dA$ where Ω = domain of the (local) $z = f(x,y)$.

$$= \iint_\Omega \sqrt{1 + \frac{F_x^2}{F_z^2} + \frac{F_y^2}{F_z^2}}\, dxdy$$

$$= \iint_\Omega \frac{\sqrt{F_x^2 + F_y^2 + F_z^2}}{|F_z|}\, dxdy$$

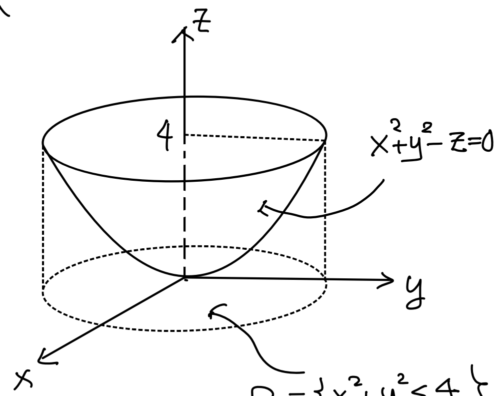
Thm 12 If $S = F^{-1}(c)$ is a <u>smooth</u> level surface such that $F_z \neq 0$, and can be represented by an implicit function over a domain Ω.

Then $$\text{Area}(S) = \iint_\Omega \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_\Omega \frac{|\vec{\nabla} F|}{|F_z|}\, dx\,dy$$

(Similar for the cases that $F_x \neq 0$ or $F_y \neq 0$)

eg 54: Find surface area of the paraboloid $x^2+y^2-z=0$ below $z=4$

(This is in fact a graph, but we do it using method of level surface)



Soln: Let $F(x,y,z) = x^2+y^2-z$

For $z=4$, $x^2+y^2-z=0 \Rightarrow x^2+y^2=4$

$\Rightarrow$ projected region $\Omega = \{(x,y) : x^2+y^2 \leq 4\}$

$\vec{\nabla} F = 2x\hat{i} + 2y\hat{j} - \hat{k}$

$\therefore F_z = -1 \neq 0, \ \forall (x,y) \in \Omega$

$$\therefore \text{Surface Area} = \iint_\Omega \frac{|\vec{\nabla} F|}{|F_z|}\, dA = \iint_\Omega \frac{\sqrt{4x^2+4y^2+1}}{|-1|}\, dx\,dy$$

(check!) $$= \iint_{\{x^2+y^2 \leq 4\}} \sqrt{4x^2+4y^2+1}\, dx\,dy = \frac{\pi}{6}\left[(\sqrt{17})^3 - 1\right]$$

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Def 16 Surface Integral (of a function)

Suppose $G: S \to \mathbb{R}$ is a continuous function on a surface S, parametrized by $\vec{r}(u,v)$, $(u,v) \in R$ (region R). Then the integral of G on S is

$$\iint_S G\, d\sigma \stackrel{\text{def}}{=\!=} \iint_R G(\vec{r}(u,v))\, |\vec{r}_u \times \vec{r}_v|\, dA$$

area element of S

element area of the parameter space $dA = du\,dv$

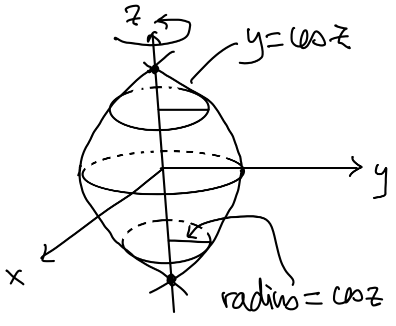
Note: In the cases of graph or level surface, we have

(i) $$\iint_S G\, d\sigma = \iint_{(x,y)} G(x,y,f(x,y))\sqrt{1+|\vec{\nabla} f|^2}\, dx\,dy$$ (for $z = f(x,y)$)

(ii) $$\iint_S G\, d\sigma = \iint_{(x,y)} G(x,y,z)\, \frac{|\vec{\nabla} F|}{|F_z|}\, dx\,dy$$ (for $F(x,y,z) = c$, $F_z \neq 0$)

(may be difficult to find these: region & z in terms of (x,y))

eg 56 (a surface of revolution of the curve $y = \cos z$)



$\left(-\frac{\pi}{2} \le z \le \frac{\pi}{2}\right)$

Let $G(x,y,z) = \sqrt{1-x^2-y^2}$ be a function on S

Find $\iint_S G\, d\sigma$.

<u>Soln</u>: S can be parametrized by

$$\begin{cases} x = \cos z \cos\theta, \\ y = \cos z \sin\theta, \\ z = z, \end{cases} \quad \begin{array}{l} \theta \in [-\pi, \pi] \\ z \in [-\frac{\pi}{2}, \frac{\pi}{2}] \end{array}$$

i.e. $\vec{r}(\theta, z) = \cos z \cos\theta\, \hat{i} + \cos z \sin\theta\, \hat{j} + z\, \hat{k}$

(Note: there is an exceptional set of "1-dim" so that $\vec{r}$ is not one-to-one, or not smooth corresponds to

$\theta = \pi$ or $-\pi$, $z = -\frac{\pi}{2}$ or $\frac{\pi}{2}$)

$$\Rightarrow \begin{cases} \vec{r}_\theta = -\cos z \sin\theta\, \hat{i} + \cos z \cos\theta\, \hat{j} \\ \vec{r}_z = -\sin z \cos\theta\, \hat{i} - \sin z \cos\theta\, \hat{j} + \hat{k} \end{cases} \quad \text{(Check!)}$$

$$\vec{r}_\theta \times \vec{r}_z = \cos z \cos\theta\, \hat{i} + \cos z \sin\theta\, \hat{j} + \sin z \cos z\, \hat{k} \quad \text{(Check!)}$$

$$\Rightarrow |\vec{r}_\theta \times \vec{r}_z| = \sqrt{\cos^2 z (1 + \sin^2 z)} = \cos z \sqrt{1 + \sin^2 z} \quad \text{(Check!)}$$

$(\cos z \geq 0 \text{ for } z \in [-\frac{\pi}{2}, \frac{\pi}{2}])$

Then
$$\iint_S G\, d\sigma = \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} G(\vec{r}(\theta, z))\, |\vec{r}_\theta \times \vec{r}_z|\, dz\, d\theta$$

$$= \int_{-\pi}^{\pi} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sqrt{1 - \cos^2 z}\, \cos z \sqrt{1 + \sin^2 z}\, dz\, d\theta$$

$$= 4\pi \int_0^{\frac{\pi}{2}} \sin z \cos z \sqrt{1 + \sin^2 z}\, dz$$

$$= \cdots = \frac{4\pi}{3}(2\sqrt{2} - 1)$$

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