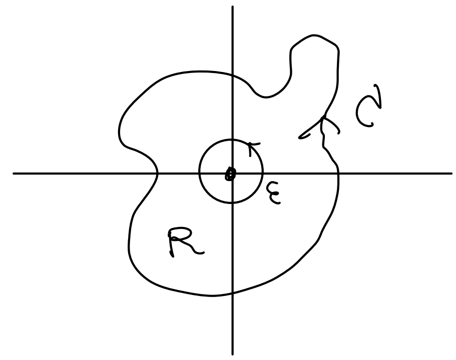
(Cont'd)

Soln (a) Recall that $\vec{\nabla}\times\vec{F} = 0$

(Note that Green's Thm doesn't apply directly, since $C$ encloses the origin $(0,0)$ where $\vec{F}$ is not defined.)

Choose $\varepsilon > 0$ small enough such that the circle $C_\varepsilon$ of radius $\varepsilon$ centered at $(0,0)$ is completely enclosed by $C$.




$\vec{F}$ is smooth in the region $R$ between $C$ and $C_\varepsilon$.

Hence the general form of Green's Thm implies

$$0 = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, dA = \oint_C \vec{F}\cdot d\vec{r} - \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r}$$

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r}$$

$$= \oint_{C_\varepsilon} \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

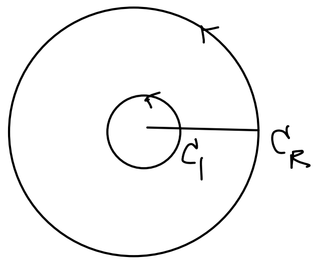
Parametrize $C_\varepsilon$ by $\begin{cases} x = \varepsilon\cos\theta \\ y = \varepsilon\sin\theta \end{cases}$, $0 \le \theta \le 2\pi$

$$\Rightarrow \oint_{C_\varepsilon} \vec{F}\cdot d\vec{r} = \int_0^{2\pi} \left[ \frac{-\cancel{\varepsilon}\sin\theta}{\cancel{\varepsilon^2}}(-\cancel{\varepsilon}\sin\theta) + \frac{\cancel{\varepsilon}\cos\theta}{\cancel{\varepsilon^2}}(\cancel{\varepsilon}\cos\theta) \right] d\theta$$

$$= 2\pi$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = 2\pi$$

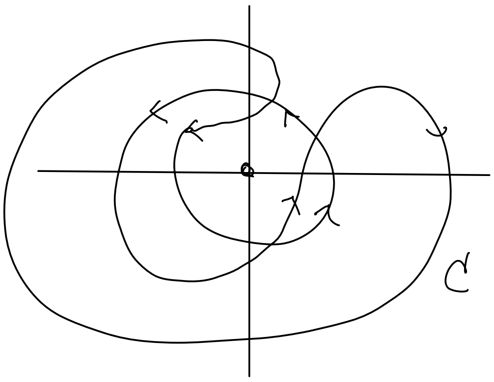
[ In fact, we've proved that $\oint_{C_R} \vec{F}\cdot d\vec{r} = 2\pi$, $\forall$ radius $R>0$, which can be seen by consider the domain between $C_1$ & $C_R$



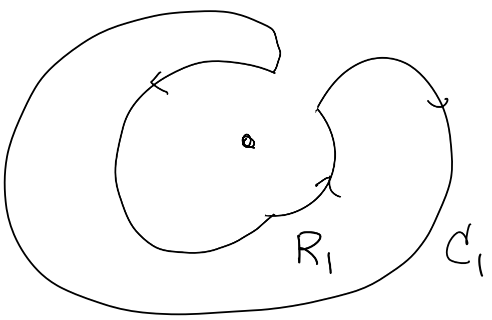

Green's Thm

$$\Rightarrow \oint_{C_1} \vec{F}\cdot d\vec{r} = \oint_{C_R} \vec{F}\cdot d\vec{r}$$ ]

(b)




Decompose the curve into




doesn't enclose $(0,0)$

$$\oint_{C_1} \vec{F}\cdot d\vec{r} = \iint_{R_1} (\vec{\nabla}\times\vec{F})\cdot\hat{k}\, dA = 0$$

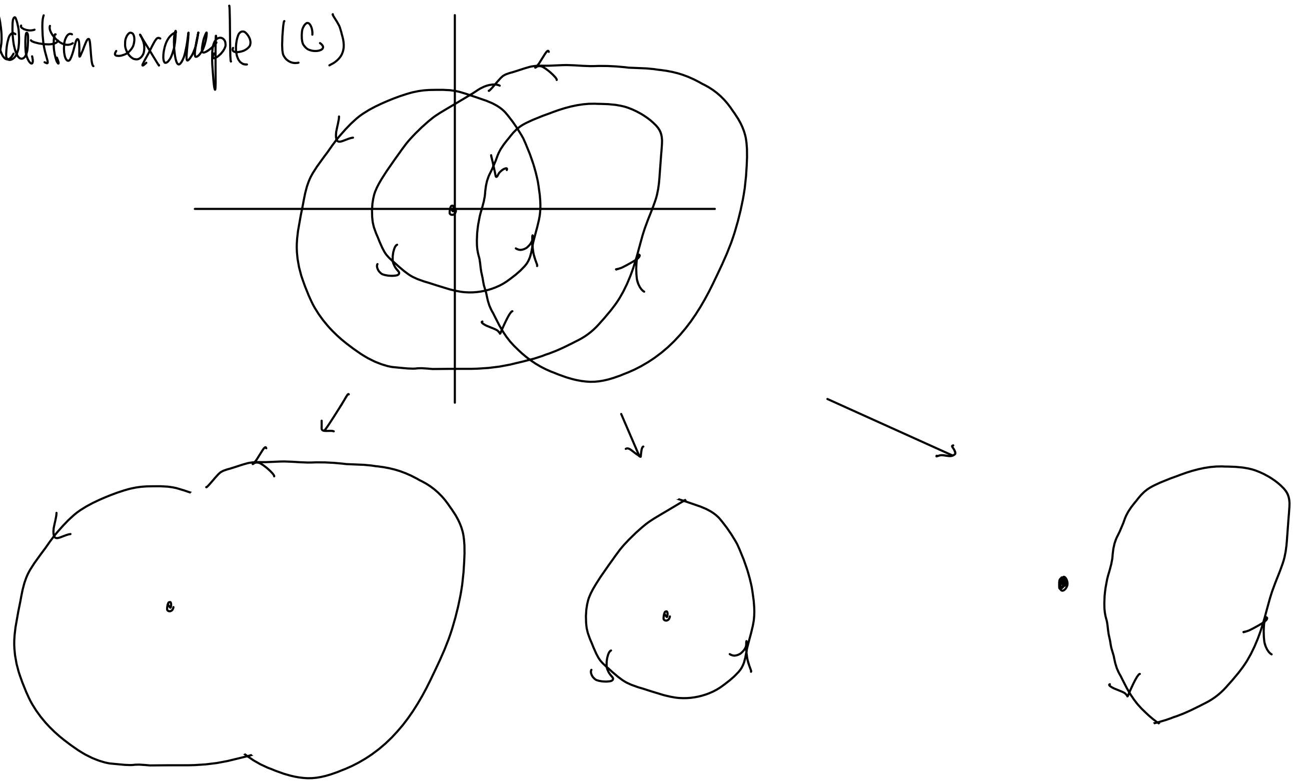

encloses $(0,0)$

$$\oint_{C_2} \vec{F}\cdot d\vec{r} = 2\pi \quad \text{(by part (a))}$$

$$\therefore \oint_C \vec{F}\cdot d\vec{r} = 0 + 2\pi = 2\pi$$

$\#$

Addition example (c)



$\oint \cdots = 2\pi$ $\qquad$ $\oint \cdots = 2\pi$ $\qquad$ $\oint \cdots = 0$

Hence $\oint_C \vec{F} \cdot d\vec{r} = 2\pi + 2\pi + 0 = 4\pi$

(optional ex! : think of some examples with $-2\pi$)

# Surface Area & Integral

**Def 14** Parametric Surface (Surface with parametrization)

A parametric surface (or a parametrization of a surface) in $\mathbb{R}^3$ is a mapping of 2-variables into $\mathbb{R}^3$:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

And it is called smooth if

(1) $\vec{r}$ is $C^1$ (i.e. $x_u, x_v, y_u, y_v, z_u, z_v$ are continuous)

(2) $\boxed{\vec{r}_u \times \vec{r}_v \neq \vec{0}}$ $\forall\, u, v$

where

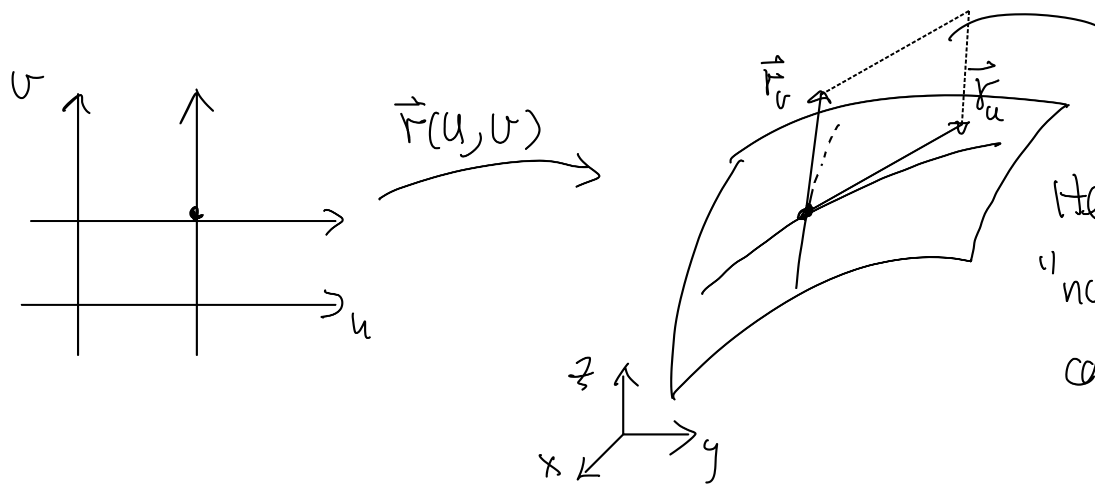
$$\begin{cases} \vec{r}_u = \dfrac{\partial \vec{r}}{\partial u} = \dfrac{\partial x}{\partial u}\hat{i} + \dfrac{\partial y}{\partial u}\hat{j} + \dfrac{\partial z}{\partial u}\hat{k} \\[2ex] \vec{r}_v = \dfrac{\partial \vec{r}}{\partial v} = \dfrac{\partial x}{\partial v}\hat{i} + \dfrac{\partial y}{\partial v}\hat{j} + \dfrac{\partial z}{\partial v}\hat{k} \end{cases}$$

$$\left( \begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases} \right)$$

Note: Condition (2) $\Rightarrow$ $\vec{r}_u, \vec{r}_v$ are linearly independent

$\Rightarrow$ span$\{\vec{r}_u, \vec{r}_v\}$ is in fact a 2-dim'l subspace.

$\Rightarrow$ "surface" cannot be degenerated to a curve or a point.

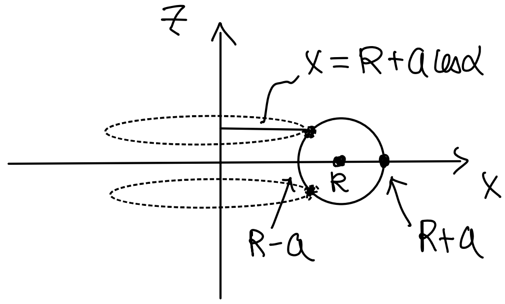



area $= |\vec{r}_u \times \vec{r}_v| \neq 0$

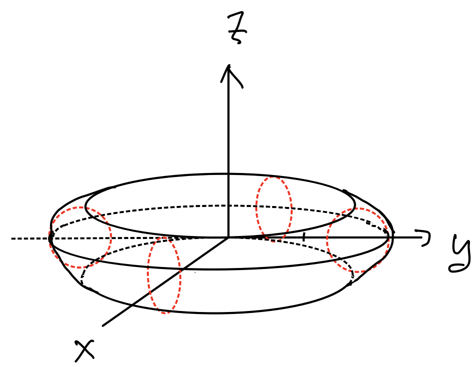
Hence, it is a "non-degenerate" condition.

<u>eg51</u> (Torus)

Consider the circle on the xz-plane (ie. $y=0$) with radius $a>0$ centered at $(x,z)=(R,0)$ with $R>a$.




A parametrization is
$$\begin{cases} x = R + a\cos\alpha \\ z = a\sin\alpha \end{cases} \qquad \alpha \in [0, 2\pi]$$




rotating the circle, we have a Torus

Then the parametrization of the Torus is
$$\begin{cases} x = (R + a\cos\alpha)\cos\theta \\ y = (R + a\cos\alpha)\sin\theta \\ z = a\sin\alpha \end{cases}, \qquad \begin{array}{l} \alpha \in [0, 2\pi] \\ \theta \in [0, 2\pi] \end{array}$$

Ex: Check that it is a smooth surface:

It is clearly $C^1$, need to check

$$(x_\alpha, y_\alpha, z_\alpha) \times (x_\theta, y_\theta, z_\theta) \neq \vec{0}$$

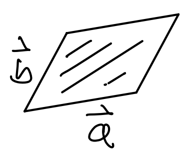
(See next example)

Note: This torus can also be described as

$$\left(\sqrt{x^2+y^2} - R\right)^2 + z^2 = a^2 \qquad (\text{Ex!})$$

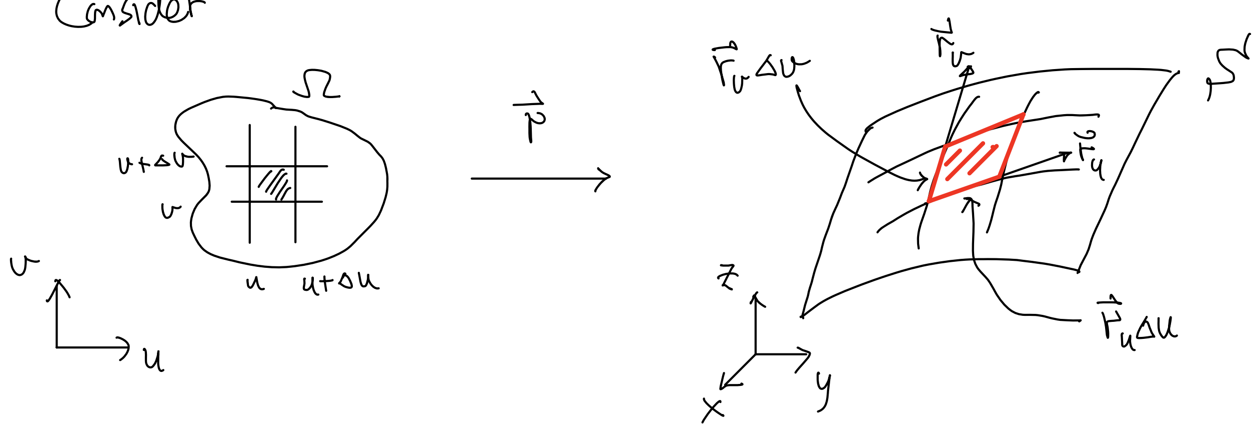
## Surface Area

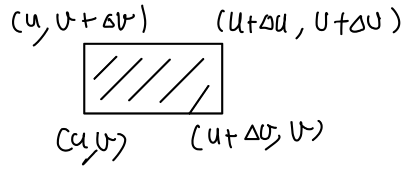
Recall: for $\vec{a}, \vec{b} \in \mathbb{R}^3$

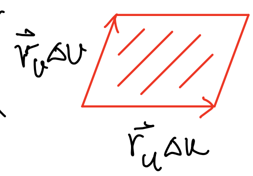
$\left|\vec{a} \times \vec{b}\right| = \text{Area}\Bigg($  $\Bigg)$

Let $\vec{r}(u,v)$ be a parametrization of a surface $S$ with $(u,v) \in \Omega$

Consider



$\Rightarrow$ "Area" on the surface corresponding to 


is approx. $=$ Area $\Big($ 

$\Big)$

$$= \left| (\vec{r}_u \Delta u) \times (\vec{r}_v \Delta v) \right|$$

$$= \left| \vec{r}_u \times \vec{r}_v \right| \Delta u \Delta v$$

Hence "Area element" of $S$, denoted by $d\sigma$, is given by

$$d\sigma = \left| \vec{r}_u \times \vec{r}_v \right| du\, dv$$

$$\boxed{d\sigma = \left| \vec{r}_u \times \vec{r}_v \right| dA}$$

area element in the $(u,v)$-space.

Therefore, we make the following

<u>Def 15</u>: Let $S \subset \mathbb{R}^3$ be a <u>smooth</u> <u>parametric</u> surface given by $\vec{r}(u,v)$ for $(u,v) \in \Omega \subset \mathbb{R}^2$. Then

$$\text{Area}(S) \overset{\text{def}}{=\!=} \iint_\Omega \left| \vec{r}_u \times \vec{r}_v \right| dA$$

$$= \iint_\Omega \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| dA$$

(i.e. $\text{Area}(S) = \iint_\Omega d\sigma$)

eg 52: Surface area of torus given by ($R > a > 0$ are constants)

$$\begin{cases} x = (R + a\cos\alpha)\cos\theta \\ y = (R + a\cos\alpha)\sin\theta \\ z = a\sin\alpha \end{cases}, \quad \begin{array}{l} \alpha \in [0, 2\pi] \\ \theta \in [0, 2\pi] \end{array}$$

Soln (i.e.)

$$\vec{r}(\alpha, \theta) = (R + a\cos\alpha)\cos\theta\,\hat{i} + (R + a\cos\alpha)\sin\theta\,\hat{j} + a\sin\alpha\,\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_\alpha = -a\sin\alpha\cos\theta\,\hat{i} - a\sin\alpha\sin\theta\,\hat{j} + a\cos\alpha\,\hat{k} \\ \vec{r}_\theta = -(R + a\cos\alpha)\sin\theta\,\hat{i} + (R + a\cos\alpha)\cos\theta\,\hat{j} \end{cases}$$

$$\& \quad \vec{r}_\alpha \times \vec{r}_\theta = -a(R + a\cos\alpha)\cos\theta\cos\alpha\,\hat{i} - a(R + a\cos\alpha)\sin\theta\cos\alpha\,\hat{j} - a(R + a\cos\alpha)\sin\alpha\,\hat{k}$$

(check!)

$$\Rightarrow |\vec{r}_\alpha \times \vec{r}_\theta| = a(R + a\cos\alpha) > 0$$

(check!)

(So, the surface is "smooth")

$$\text{Area (Torus)} = \iint_\Omega |\vec{r}_\alpha \times \vec{r}_\theta|\, dA$$

$$= \int_0^{2\pi}\int_0^{2\pi} a(R + a\cos\alpha)\, d\alpha\, d\theta$$

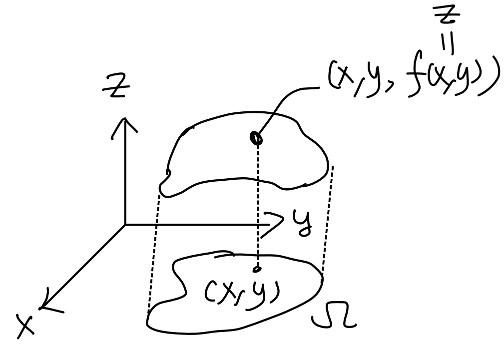
$$= 4\pi^2 Ra$$

(check!)

#

## Surface area of a graph

$z = f(x,y), \quad (x,y) \in \Omega$




Choose the following "natural" parametrization of the graph

$$\vec{r}(x,y) = x\hat{i} + y\hat{j} + f(x,y)\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_x = \hat{i} + f_x\hat{k} \\ \vec{r}_y = \hat{j} + f_y\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_x \times \vec{r}_y = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 0 & f_x \\ 0 & 1 & f_y \end{vmatrix} = -f_x\hat{i} - f_y\hat{j} + \hat{k}$$

$$\Rightarrow \quad |\vec{r}_x \times \vec{r}_y| = \sqrt{f_x^2 + f_y^2 + 1} = \sqrt{1 + |\vec{\nabla} f|^2} \geq 1$$ (non-zero, hence "smooth" if $f \in C^1$)

---

**Thm 11**: The surface area of a $C^1$ graph $S$ given by

$$z = f(x,y), \quad (x,y) \in \Omega \subset \mathbb{R}^2$$

is

$$\text{Area}(S) = \iint_\Omega \sqrt{1 + |\vec{\nabla} f|^2}\, dA = \iint_\Omega \sqrt{1 + f_x^2 + f_y^2}\, dA$$

---

( Similarly for $x = f(y,z)$ or $y = f(x,z)$ )