In these notation, the Green's thm can be written as

$$\boxed{\begin{array}{l}
\underline{\text{Vector form of Green's Thm}} \\[1em]
\underline{\text{normal form}} \qquad \boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \operatorname{div}\vec{F}\,dA} \\[1em]
\qquad\qquad\quad \text{or} \qquad \boxed{\oint_C \vec{F}\cdot\hat{n}\,ds = \iint_R \vec{\nabla}\cdot\vec{F}\,dA} \\[1em]
\underline{\text{tangential form}} \qquad \boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R \operatorname{curl}\vec{F}\cdot\hat{k}\,dA} \\[1em]
\qquad\qquad\quad \text{or} \qquad \boxed{\oint_C \vec{F}\cdot\hat{T}\,ds = \iint_R (\vec{\nabla}\times\vec{F})\cdot\hat{k}\,dA}
\end{array}}$$

And Thm 10 can be written as

$$\boxed{\begin{array}{l}
\underline{\text{Thm } 10'}:\ \Omega\ \underline{\text{simply-connected}}\ \&\ \underline{\text{connected}},\ \vec{F}\in C^1. \\
\qquad \text{Then}\quad \vec{F} = \text{conservative} \iff \operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F} = 0
\end{array}}$$ (check)

<u>Note</u>: (i) $\operatorname{curl}\vec{F} = \vec{\nabla}\times\vec{F}$ defined only in $\mathbb{R}^3$ $(\supset \mathbb{R}^2)$

(ii) but $\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F}$ can be defined on $\mathbb{R}^n$ for any n

In particular, in $\mathbb{R}^3$

$$\boxed{\begin{array}{l}
\underline{\text{Def } 12'}\ \text{ The}\ \underline{\text{divergence}}\ \text{of } \vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \text{ is defined to be} \\
\operatorname{div}\vec{F} = \vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(M\hat{i} + N\hat{j} + L\hat{k}\right) = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial L}{\partial z}
\end{array}}$$

Then one can easily check the following facts : (Ex!)

$$\boxed{\begin{array}{l} \text{(i)}\ \vec{\nabla}\times(\vec{\nabla} f) = 0 \quad (\text{i.e. curl } \vec{\nabla} f = 0) \\ \text{(ii)}\ \vec{F} \text{ conservative} \Rightarrow \text{curl } \vec{F} = \vec{\nabla}\times\vec{F} = 0 \\ \text{(iii)}\ \vec{\nabla}\cdot(\vec{\nabla}\times\vec{F}) = 0 \quad (\text{i.e. div}(\text{curl } \vec{F}) = 0) \end{array}}$$

<u>Remark</u> : $\vec{\nabla}\cdot(\vec{\nabla} f) \neq 0$ in general, and it is called the <u>Laplacian</u> of f and is denoted by

$$\boxed{\begin{aligned} \vec{\nabla}^2 f &= \vec{\nabla}\cdot(\vec{\nabla} f) = \text{div}(\vec{\nabla} f) \\ &= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \end{aligned}}$$

[In graduate level, it will be denoted by $\Delta = \vec{\nabla}^2$ or $\Delta = -\vec{\nabla}^2$]

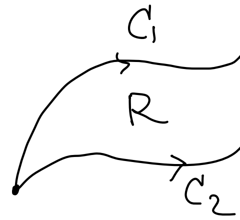
The "operator" $\vec{\nabla}^2$ is called the <u>Laplace operator</u> and the equation $\vec{\nabla}^2 f = 0$ is called the <u>Laplace equation</u>. Solutions to the Laplace equation are called <u>harmonic functions</u>.

Pf of Thm 10 ($n=2$)

We only need to show that if Ω is simply-connected (& connected)

and $\vec{\nabla} \times \vec{F} = 0$ (i.e. $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$)

then $\vec{F}$ is conservative.



Case 1 C_1, C_2 have no intersection (except at the same end points)

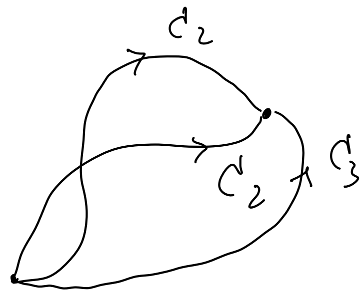
Then "Ω is simple connected"

$\Rightarrow$ the region R enclosed by C_1 & C_2 lies completely inside Ω.

Then Green's Thm $\Rightarrow$

$$0 = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \pm \oint_{C_1 - C_2} (M\,dx + N\,dy)$$

$$\Rightarrow \int_{C_1} (M\,dx + N\,dy) = \int_{C_2} (M\,dx + N\,dy)$$



Case 2 C_1 & C_2 intersect

Pick another curve C_3 with the same starting point and end point, and does not intersect C_1 or C_2.

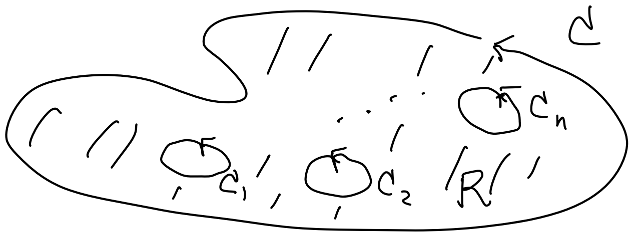
Then Case 1 $\Rightarrow$ $\int_{C_1} M\,dx + N\,dy = \int_{C_3} M\,dx + N\,dy = \int_{C_2} M\,dx + N\,dy$

$\therefore \int_C \vec{F} \cdot d\vec{r}$ is indep. of path & hence $\vec{F}$ is conservative. #

In order to apply Green's Thm to more general situations, we need

Thm (Green's Thm (general form))

Suppose that we have a simple closed curve C in $\mathbb{R}^2$



Suppose that $C_1, C_2, \cdots, C_n$ be pairwise disjoint, piecewise smooth, simple closed curves, such that $C_1, \cdots, C_n$ are enclosed by C.

(All $C, C_1, \cdots, C_n$ are anti-clockwise oriented.)

Let R be the region between C and $C_1, \cdots, C_n$.

Suppose that $\vec{F} = M\hat{i} + N\hat{j}$ is defined on some open set containing R, and is C^1. Then

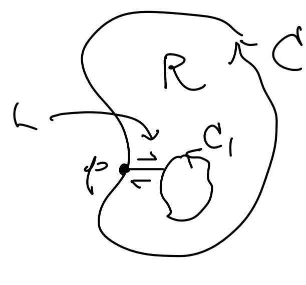
$$\boxed{\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \oint_C M\,dx + N\,dy - \sum_{i=1}^{n} \oint_{C_i} M\,dx + N\,dy}$$

(This is the tangential form. The normal form is similar)

Sketch of Proof

For simplicity, only one C_1 inside C

We connect C & C_1 by an "arc" L



and conside the "simple" closed curve

(starting from the point p) $C^* = C + L - C_1 - L$

Then the region R enclosed between C & C_2 is the region enclosed by C^* except the arc L.

Hence $$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \iint_{R \setminus L} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \oint_{C^*} (M\,dx + N\,dy) \quad \text{(by Green's)}$$

$$= \left(\oint_C + \int_L - \oint_{C_1} - \int_L\right)(M\,dx + N\,dy)$$

$$= \oint_C M\,dx + N\,dy - \oint_{C_1} M\,dx + N\,dy$$

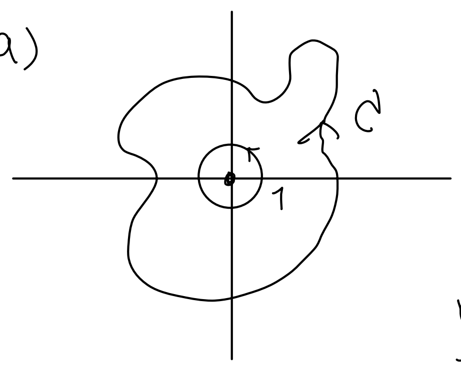
#

<u>eg 49</u>: $\vec{F} = \dfrac{-y}{x^2+y^2}\hat{i} + \dfrac{x}{x^2+y^2}\hat{j}$ on $\mathbb{R}^2 \setminus \{(0,0)\} = \Omega$

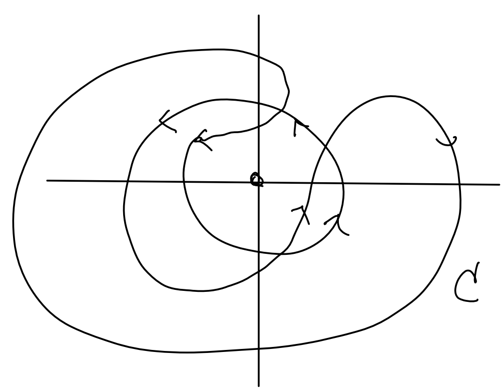
we've calculated $\oint_{C_1} \vec{F} \cdot d\vec{r} = 2\pi$ for $C_1 : x^2+y^2 = 1$ (anti-clockwise)

How about

(a)



(b)



$\oint_C \vec{F} \cdot d\vec{r} = ?$