eg 48 Verify both forms of Green's Thm for

$$\vec{F}(x,y) = (x-y)\hat{i} + x\hat{j} \quad \text{on } \Omega = \mathbb{R}^2, \text{ is } C^\infty.$$

$$C = \text{unit circle} = \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [0, 2\pi]$$

Then R = region enclosed by $C = \{x^2+y^2<1\}$ the unit disc.

(We also write $C = \partial R$ boundary of R)

Soln $\quad M = x-y \quad \& \quad N = x$

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1 \quad ; \quad \frac{\partial N}{\partial x} = 1, \quad \frac{\partial N}{\partial y} = 0$$

On C, $x = \cos t$, $y = \sin t$, $t \in [0, 2\pi]$

Normal form

$$\text{L.H.S.} = \oint_C M\,dy - N\,dx$$

$$= \int_0^{2\pi} (\cos t - \sin t)\,d\sin t - \cos t\,d\cos t$$

$$= \int_0^{2\pi} \cos^2 t\,dt = \pi \qquad \text{(check!)}$$

$$\text{R.H.S.} = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \iint_R (1+0)\,dx\,dy = \iint_R dx\,dy = \pi$$

Tangential form $\quad \text{L.H.S.} = \oint_C M\,dx + N\,dy = \cdots = 2\pi \quad$ (check!)

$$\text{R.H.S.} = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \iint_R (1-(-1))\,dx\,dy = 2\pi$$

(Note: This example shows that even the 2 forms are equivalent, the values involved may differ.)

Pf of Green's Thm (tangential form)

Recall: A region R is of special type:

type (1): If $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

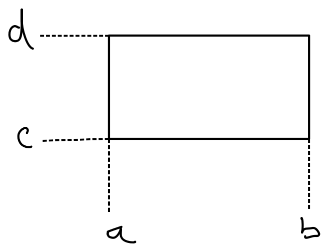
for some continuous functions $g_1(x)$ & $g_2(x)$.

type (2): If $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

for some continuous functions $h_1(x)$ & $h_2(x)$.

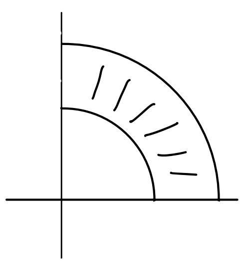
Now: If R is both type (1) and type (2), it said to be simple.

eg 4.8: (i)



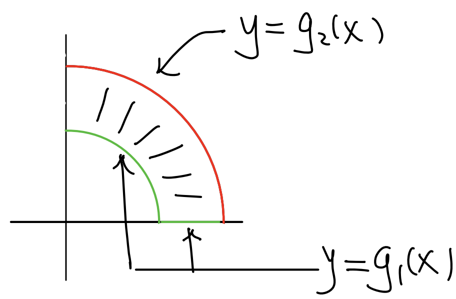
rectangle is simple

(ii)

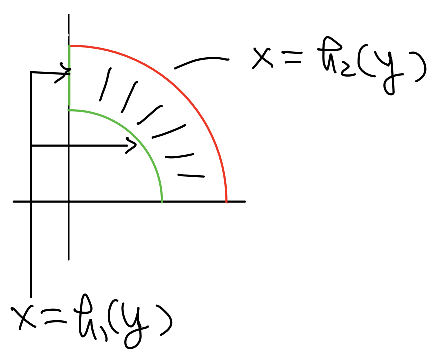


simple

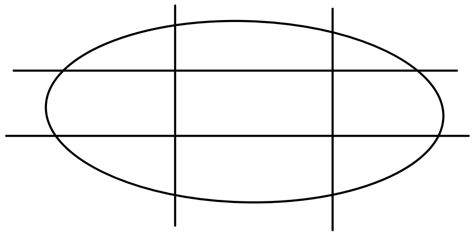
type (1)



type (2)



(iii')



2 intersections at most

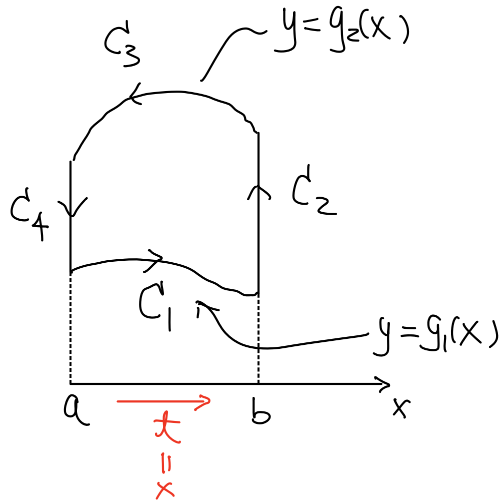
2 intersections at most

$\Rightarrow$ simple

$$\forall a \in \mathbb{R}: \quad \left.\begin{array}{l} \#\{\partial R \cap \{x=a\}\} \leq 2 \\ \#\{\partial R \cap \{y=a\}\} \leq 2 \end{array}\right\} \Rightarrow \text{simple}$$

(provided ∂R is piecewise smooth)

(Proof: Omitted)

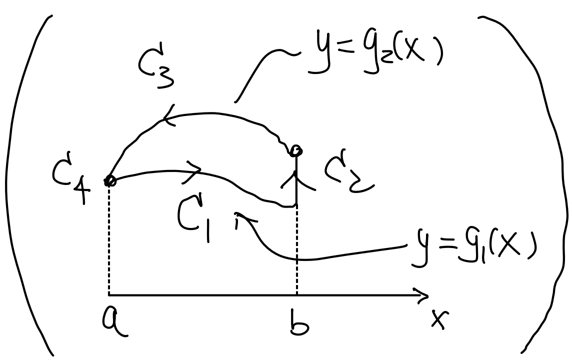


Pf of Green's Thm for Simple Region

By definition, R is of type (I) and can be written as

$$R=\{(x,y): a \leq x \leq b,\ g_1(x) \leq y \leq g_2(x)\}$$

Let denote the components of the boundary of R by C_1, C_2, C_3 and C_4 as in the figure (Note: C_2 and/or C_4 could just be a point)



Then $\partial R = C_1 + C_2 + C_3 + C_4$ as oriented curve.

(using "+" instead of "$\cup$" to denote the orientation)

Now $C_1 : \{y = g_1(x)\}$ can be parametrized by

$(x,y) : \vec{r}(t) = (t, g_1(t))$, $a \leq t \leq b$ (with correct orientation)

$$\therefore \int_{C_1} M\,dx = \int_a^b M(t, g_1(t))\,dt$$

Similarly "$-C_3$" can be parametrized by

$\vec{r}(t) = (t, g_2(t))$, $a \leq t \leq b$ (with correct orientation)

$$\therefore \int_{-C_3} M\,dx = \int_a^b M(t, g_2(t))\,dt$$

For $C_2 : \{x = b, \ g_1(b) \leq y \leq g_2(b)\}$, it can be parametrized by

$\vec{r}(t) = (b, t)$, $g_1(b) \leq t \leq g_2(b)$ (with correct orientation)

$$\therefore \int_{C_2} M\,dx = 0 \qquad \left(\text{since } \frac{dx}{dt} = 0\right)$$

Similarly $\int_{C_4} M\,dx = -\int_{-C_4} M\,dx = 0$.

Hence
$$\oint_{\partial R} M\,dx = \sum_{i=1}^{4} \oint_{C_i} M\,dx$$

$$= \int_a^b [M(t, g_1(t)) - M(t, g_2(t))]\,dt$$

$$\left(= \int_a^b [M(x, g_1(x)) - M(x, g_2(x))]\,dx \right)$$

On the other hand, Fubini's Thm $\Rightarrow$

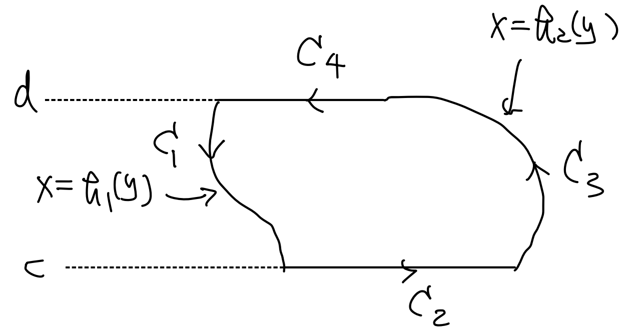
$$\iint_R -\frac{\partial M}{\partial y}\, dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} -\frac{\partial M}{\partial y}\, dy \right) dx$$

$$= \int_a^b \left[M(x, g_1(x)) - M(x, g_2(x)) \right] dx$$

$$= \oint_{\partial R} M\, dx$$

Similar, R is also type (2),

$R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$



$$\oint_{\partial R} N\, dy = -\int_c^d N(h_1(t), t)\, dt + 0 + \int_c^d N(h_2(t), t)\, dt + 0$$

$$= \int_c^d \left[N(h_2(t), t) - N(h_1(t), t) \right] dt$$

$$= \int_c^d \left[\int_{h_1(y)}^{h_2(y)} \frac{\partial N}{\partial x}\, dx \right] dy$$

$$= \iint_R \frac{\partial N}{\partial x}\, dA$$

All together

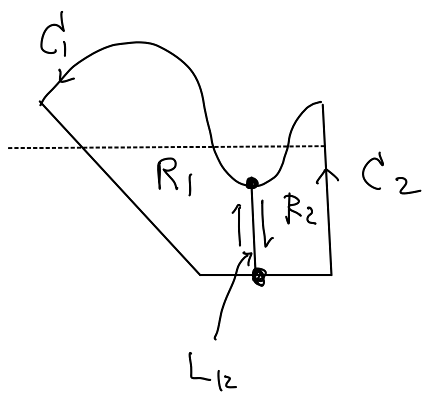
$$\oint_{\partial R} (M\,dx + N\,dy) = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

$\#$

Proof of Green's Thm for

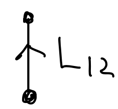
$R =$ finite union of simple regions with intersections only along some boundary line segments, and those line segments touch only at the end points at most.

eg:



R_1, R_2 are simple but $R = R_1 \cup R_2 \neq$ simple

$\partial R_1 = C_1 + L_{12}$



$\partial R_2 = C_2 - L_{12}$

with anti-clockwise orientation

and $\partial R = C_1 + C_2$

By assumption $R = \cup R_i$ finite union s.t.

- R_i are simple, and
- $R_i \cap R_j =$ line segment of a common boundary portion denoted by

  $L_{ij} \quad (i \neq j)$

Then $\displaystyle\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \iint_{R_i} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$

$\displaystyle = \sum_i \oint_{\partial R_i} M\,dx + N\,dy$ (by Green's Thm for simple region)

Denote C_i = the part of ∂R_i with no intersection with any other R_j (except at the end points)

Then $$\partial R_i = C_i + \sum_{\substack{j \\ (j \neq i)}} L_{ij}$$

where L_{ij} is oriented according to the anti-clockwise orientation of ∂R_i

Hence

$$\iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA = \sum_i \oint_{C_i + \sum_{\substack{j \\ (j\neq i)}} L_{ij}} M dx + N dy$$

$$= \sum_i \int_{C_i} M dx + N dy + \sum_i \sum_{\substack{j \\ (j \neq i)}} \int_{L_{ij}} M dx + N dy$$

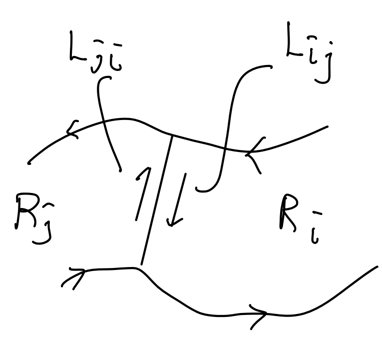
Note that, as C_i is not a common boundary of any other R_j,

$$\sum_i C_i = \partial R$$

$$\therefore \sum_i \int_{C_i} M dx + N dy = \oint_{\partial R} M dx + N dy$$

Finally, we have $L_{ji} = -L_{ij}$

as R_i & R_j are located on the two different sides of the



common boundary.

$$\sum_{i} \sum_{\substack{j \\ (j \neq i)}} \int_{L_{ij}} M dx + N dy = \sum_{\substack{i, j \\ (i \neq j)}} \int_{L_{ij}} M dx + N dy$$

$$= \sum_{i<j} \int_{L_{ij}} M dx + N dy + \sum_{\substack{\cancel{j<i} \\ i<j}} \int_{\substack{L_{\cancel{ij}} \\ ji}} M dx + N dy$$

$$= \sum_{i<j} \left(\int_{L_{ij}} M dx + N dy + \int_{L_{ji}} M dx + N dy \right)$$

$$= \sum_{i<j} \left(\int_{L_{ij}} M dx + N dy - \int_{L_{ij}} M dx + N dy \right)$$

$$= 0$$

This 2nd case basically include almost all situations in the level of Advanced Calculus.

The proof of <u>general case</u> needs "analysis" and will be omitted here.

※

$\boxed{\begin{array}{c}\text{\underline{Def 12}: The \underline{divergence} of } \vec{F} = M\hat{i} + N\hat{j} \text{ is defined to be} \\ \operatorname{div}\vec{F} = \dfrac{\partial M}{\partial x} + \dfrac{\partial N}{\partial y}\end{array}}$

<u>Note</u>: $$\operatorname{div}\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{n}\, ds$$

$\underset{=}{\text{called}}$ "flux density".

<u>Notation</u>: For $f(x,y)$, $\vec{\nabla} f = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j}$ (gradient)

$$= \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) f$$

It is convenient to denote

$$\vec{\nabla} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right)$$

Then $$\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}\right) \cdot \left(M\hat{i} + N\hat{j}\right)$$

$$= \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} = \operatorname{div}\vec{F}$$

Hence we also write

$$\boxed{\operatorname{div}\vec{F} = \vec{\nabla} \cdot \vec{F}}$$

Def 13: Define rot $\vec{F}$ to be

$$\boxed{\text{rot}\,\vec{F} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \qquad \left(\text{for } \vec{F} = M\hat{i} + N\hat{j}\right)}$$

Note: $$\text{rot}\,\vec{F} = \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \iint_{\bar{D}_\varepsilon(x,y)} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

$$= \lim_{\varepsilon \to 0} \frac{1}{\text{Area}(\bar{D}_\varepsilon(x,y))} \oint_{\partial \bar{D}_\varepsilon(x,y)} \vec{F} \cdot \hat{T}\, ds$$

$\overset{\text{called}}{=}$ circulation density

Using $\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y}$, we can write

$$\boxed{\text{rot}\,\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}}$$

Since $\vec{F} = M\hat{i} + N\hat{j} + \underset{\text{number zero}}{0} \cdot \hat{k}$ (in $\mathbb{R}^3$) $\quad (M = M(x,y)$ & $N = N(x,y))$

$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}$ (in $\mathbb{R}^3$) $\quad \left(\frac{\partial M}{\partial z} = \frac{\partial N}{\partial z} = 0\right)$

$$\Rightarrow \vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & 0 \end{vmatrix} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ M & N \end{vmatrix} \hat{k} = \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\hat{k}$$

(check!)

$\Rightarrow$ rot $\vec{F} = (\vec{\nabla} \times \vec{F}) \cdot \hat{k}$ i.e. $\hat{k}$-component of $\underline{\vec{\nabla} \times \vec{F}}$.

A name for $\vec{\nabla} \times \vec{F}$ is $\underline{\text{curl}\,\vec{F}}$: $\boxed{\text{curl}\,\vec{F} \overset{\text{def}}{=} \vec{\nabla} \times \vec{F}}$