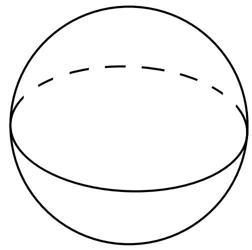
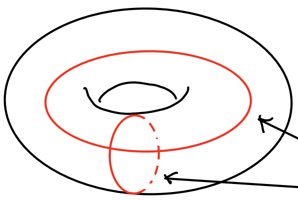
eg44 Ω_1 in eg43 is simply-connected, but Ω_2 is not simply-connected.

eg45:  $S^2 \subset \mathbb{R}^3$ $S^2 = \{x^2+y^2+z^2=1\}$ (unit sphere) is simply-connected.

eg46: 

torus $\mathbb{T}^2 \cong S^1 \times S^1 \subset \mathbb{R}^3$ is not simply-connected

those 2 closed curves cannot be contracted to a point on $\mathbb{T}^2$.

Remark: Simply-connectedness is a global condition to guarantee " "PDEs in Cor to Thm 9" $\Rightarrow$ "conservative" "

Thm 10: Suppose $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, is connected and simply-connected. Let $\vec{F}$ be C^1 vector field on Ω. Then

$\vec{F}$ is conservative on Ω $\Longleftrightarrow$ components of $\vec{F}$ satisfy the system of PDEs in the Cor to Thm 9.

(Pf: later)

eg 47 : Let $\Omega \equiv \mathbb{R}^3$ (connected and simply-connected)

Let $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$= (y+e^z)\hat{i} + (x+1)\hat{j} + (1+xe^z)\hat{k}.$$

Find the potential function f of $\vec{F}$ i.e. $\vec{\nabla} f = \vec{F}$.

Soln: That is, we want to solve

$$\frac{\partial f}{\partial x} = M, \quad \frac{\partial f}{\partial y} = N, \quad \frac{\partial f}{\partial z} = L$$

Checking M, N, L satisfy the system of PDEs in the Cor to Thm 9:

$$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial M}{\partial z} = e^z$$

$$\frac{\partial N}{\partial x} = 1 \qquad \frac{\partial N}{\partial z} = 0$$

$$\frac{\partial L}{\partial x} = e^z \qquad \frac{\partial L}{\partial y} = 0$$

(No need to check $\frac{\partial M}{\partial x}, \frac{\partial N}{\partial y}, \frac{\partial L}{\partial z}$)

Thm 10 $\Rightarrow$ existence of potential function f

To find f explicitly

$$\frac{\partial f}{\partial x} = M = y + e^z$$

$$\Rightarrow \quad f = \int (y+e^z)\,dx = x(y+e^z) + \text{"constant in } x\text{"}$$

$$= x(y+e^z) + g(y,z) \quad \text{for some function } g(y,z)$$

Then take $\frac{\partial}{\partial y}$:

$$N = x+1 = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left[x(y+e^z) + g(y,z)\right]$$

$$= x + \frac{\partial g}{\partial y}$$

$$\Rightarrow \quad \frac{\partial g}{\partial y} = 1$$

$$\Rightarrow \quad g = \int 1\, dy = y + \text{"const in } y\text{"}$$

$$= y + h(z) \quad \text{for some function } h(z).$$

$$\Rightarrow \quad f = x(y+e^z) + y + h(z)$$

Then take $\frac{\partial}{\partial z}$:

$$L = 1 + xe^z = \frac{\partial f}{\partial z} = \frac{\partial}{\partial z}\left[x(y+e^z) + y + h(z)\right]$$

$$= xe^z + h'$$

$$\Rightarrow \quad h'(z) = 1 \quad \Rightarrow \quad h(z) = z + \text{const}.$$

Hence $f(x,y,z) = x(y+e^z) + y + z + C$, where C is a constant,

is a potential function of $\vec{F}$ ※

<u>Remark</u>: To prove Thm 10 in $\mathbb{R}^2$, we need the <u>Green's Thm</u>

(in $\mathbb{R}^3$, we need the <u>Stokes' Thm</u>)

<u>Thm 11</u> (<u>Green's Theorem</u>)

Let $\Omega \subseteq \mathbb{R}^2$ be open, $\vec{F} = M\hat{i} + N\hat{j}$ be C^1 vector field on Ω;

C be a piecewise <u>"smooth"</u> <u>simple</u> <u>closed</u> <u>anti-clockwise oriented</u> curve enclosing a region R which <u>lies entirely in Ω</u>.

Then
- <u>Normal Form</u>

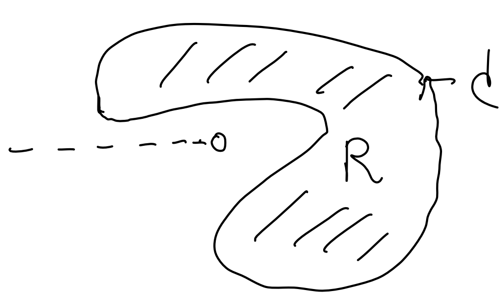
$$\oint_C \vec{F}\cdot\hat{n}\,ds = \boxed{\oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy}$$

- <u>Tangential Form</u>

$$\oint_C \vec{F}\cdot\hat{T}\,ds = \boxed{\oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy}$$

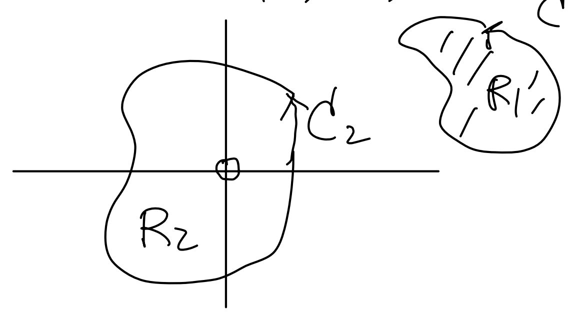
(Remark: The two forms are equivalent)

<u>Note</u>: $\Omega_1 = \mathbb{R}^2 \setminus \{x \le 0\}$



Green's Thm applies, since $R \subset \Omega_1$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$



Green's Thm applies, since $R_1 \subset \Omega_2$

$(0,0) \in R_2$, but $(0,0) \notin \Omega_2$

$\Rightarrow R_2 \not\subset \Omega_2$,

Green's Thm doesn't apply.