eg 42: Show that $\vec{F}(x,y) = \hat{i} + x\hat{j}$ is not conservative in $\mathbb{R}^2$.

Soln: ($\vec{F} \in C^\infty$) $\begin{cases} M = 1 \\ N = x \end{cases} \Rightarrow \begin{cases} \frac{\partial M}{\partial y} = 0 \\ \quad \# \\ \frac{\partial N}{\partial x} = 1 \end{cases}$

By Cor. to Thm 9, $\vec{F}$ is not conservative. ※

Remark (Important)

For a C^1 vector field $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$

$$\boxed{\vec{F} \text{ conservative}} \underset{?}{\overset{\text{Cor. to Thm 9}}{\rightleftarrows}} \boxed{\begin{array}{l} M, N, L \text{ satisfy the system} \\ \text{of PDEs in the Cor to Thm 9} \end{array}}$$

Answer: NOT TRUE in general, needs extra condition on the domain Ω ("connected" is not enough)

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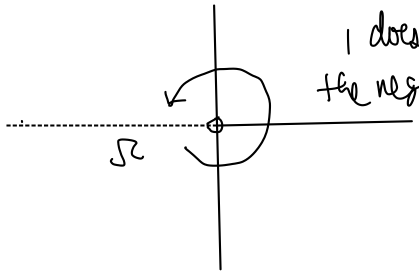
eg 43 Consider the vector field

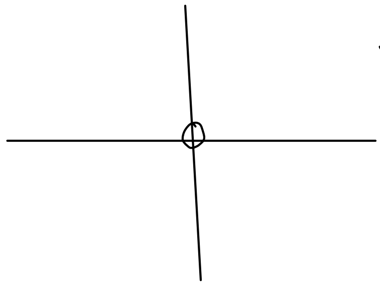
$$\vec{F} = \frac{-y}{x^2+y^2}\hat{i} + \frac{x}{x^2+y^2}\hat{j}$$

and the domains $\Omega_1 = \mathbb{R}^2 \setminus \{(x,0) \in \mathbb{R}^2 : x \leq 0\}$

$\Omega_2 = \mathbb{R}^2 \setminus \{(0,0)\}$



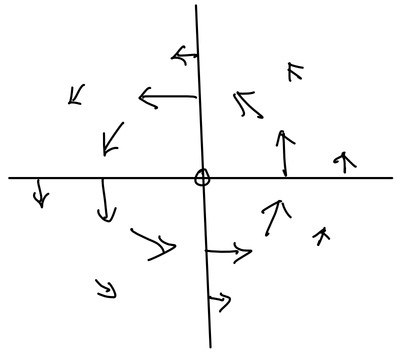
1 doesn't include the negative X-axis (and origin)



Ω_2 doesn't include the origin (but include the negative X-axis.)

In polar coordinates

$$\vec{F} = -\frac{\sin\theta}{r}\,\hat{i} + \frac{\cos\theta}{r}\,\hat{j}$$



$$\Rightarrow \begin{cases} \bullet\ \vec{F} \text{ rotates around the origin anti-clockwisely} \\ \bullet\ |\vec{F}| = \frac{1}{r} \to 0 \quad \text{as } r \to +\infty \\ \bullet\ |\vec{F}| = \frac{1}{r} \to +\infty \quad \text{as } r \to 0 \end{cases}$$

$\Rightarrow$ $\vec{F}$ cannot be extended to a C^1 vector field on $\mathbb{R}^2$.

Besides $(0,0)$, $\vec{F}$ is C^1, hence

$\vec{F}$ is C^1 on Ω_1, and also

$\vec{F}$ is C^1 on Ω_2.

Questions : Is $\vec{F}$ conservative on Ω_1 ?

Is $\vec{F}$ conservative on Ω_2 ?

<u>Soln</u>: (1) For Ω_1, and $(x,y)\in\Omega_1$ can be expressed in polar coordinates

by $\begin{cases} r>0 \\ -\pi<\theta<\pi \end{cases}$ $\quad \big((x,y)=(r\cos\theta, r\sin\theta) \big)$

straight inequalities

$\Rightarrow$ doesn't include <u>negative x-axis</u>.

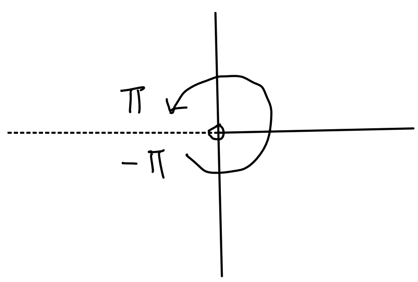
Define $f(x,y)=\theta$ "smooth" on Ω_1

Then
$$\begin{cases} \dfrac{\partial f}{\partial x} = \dfrac{\partial \theta}{\partial x} = -\dfrac{\sin\theta}{r} \\[2ex] \dfrac{\partial f}{\partial y} = \dfrac{\partial \theta}{\partial y} = \dfrac{\cos\theta}{r} \end{cases}$$
(check!)

$$\Rightarrow \quad \vec{F} = -\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j} = \frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} = \vec{\nabla} f \quad \text{on } \Omega_1$$

$\Rightarrow$ $\vec{F}$ conservative on Ω_1.

(2) For Ω_2, the "function" $f(x,y)=\theta$ <u>cannot be extended</u> to a "smooth" function on (the whole) Ω_2:



the "function" $f=\theta$ "jumps" at the negative x-axis

$\Rightarrow$ f cannot be extended to a continuous function across negative x-axis.

To show that $\vec{F}$ is not conservative on Ω_2, we consider a closed curve

$C: \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}, \quad t \in [-\pi, \pi]$

(unit circle in Ω_2, but it is not a curve in Ω_1)

Then $\oint_C \vec{F} \cdot d\vec{r} = \int_{-\pi}^{\pi} \left(-\frac{\sin\theta}{r}\hat{i} + \frac{\cos\theta}{r}\hat{j} \right) \cdot \vec{r}\,'(t)\,dt$

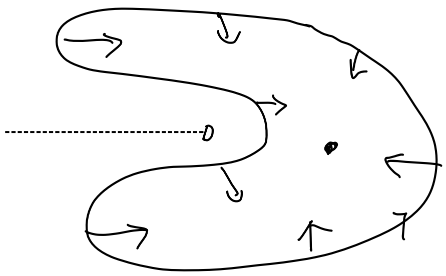
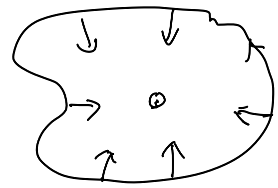
$\left((r\cos\theta, r\sin\theta) = (\cos t, \sin t) \right.$
ie $\left. r = 1, \theta = t \right)$

$= \int_{-\pi}^{\pi} (-\sin t\,\hat{i} + \cos t\,\hat{j}) \cdot (-\sin t\,\hat{i} + \cos t\,\hat{j})\,dt$

$= \int_{-\pi}^{\pi} dt = 2\pi \neq 0$.

By Thm 9, $\vec{F}$ is not conservative on Ω_2. ##

Summary

Ω_1	Ω_2
$f(x,y)=\theta$ smooth function on Ω_1	$f(x,y)=\theta$ is not a smooth function on Ω_2 (θ cannot be well-defined on the whole Ω_2)
$C: x^2+y^2=1$ is not a curve in Ω_1 ($(-1,0)\in C$ but $(-1,0)\notin\Omega_1$)	$C: x^2+y^2=1$ is a closed curve in Ω_2
 closed curve cannot circle around the origin $\Rightarrow$ closed curves can be deformed continuously (within Ω_1) to a point (in Ω_1)	 C C encloses the "hole" $\Rightarrow$ C cannot be deformed continuously (within Ω_2) to a point (in Ω_2)

Def 15 A subset $\Omega\subset\mathbb{R}^n$, $n=2$ or 3, is called simply-connected if every closed curve in Ω can be contracted to a point in Ω without ever leaving Ω.

(contracted := deformed continuously)