Conservative Vector Field

Def 14: Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open. A vector field $\vec{F}$ defined on Ω is said to be conservative if

$$\int_C \vec{F} \cdot \hat{T}\, ds \left(= \int_C \vec{F} \cdot d\vec{r} \right)$$ along an oriented curve C in Ω

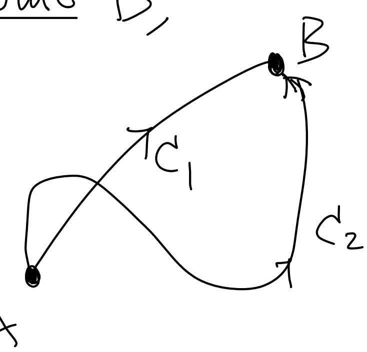
depends only on the starting point and end point of C.

Note: This is usually referred as "path independent".

i.e. If C_1 & C_2 are oriented curves with the same starting point A and end point B,

then

$$\int_{C_1} \vec{F} \cdot \hat{T}\, ds = \int_{C_2} \vec{F} \cdot \hat{T}\, ds$$



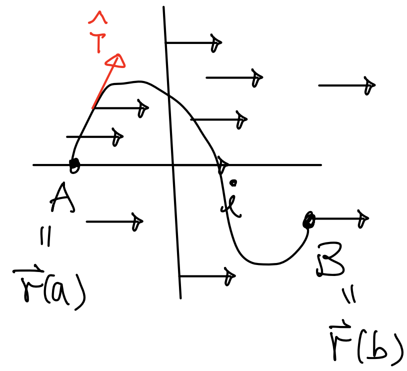
(so the value only depends on the points A & B (& direction))

Notation: If $\vec{F}$ is conservative, we sometimes write

$$\boxed{\int_A^B \vec{F} \cdot \hat{T}\, ds}$$ to denote the common value of $\int_C \vec{F} \cdot \hat{T}\, ds$ along any oriented curve C from A to B.

eg 41: $\vec{F} \equiv \hat{i}$ on $\mathbb{R}^2$

$C: \vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}, \quad a \le t \le b$



Then $\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$

$$= \int_a^b \hat{i} \cdot \left(x'(t)\hat{i} + y'(t)\hat{j} \right) dt$$

$$= \int_a^b x'(t)\, dt = x(b) - x(a)$$

x-coordinates at $\vec{r}(b)$ & $\vec{r}(a)$ respectively

$\therefore \int_C \vec{F} \cdot \hat{T}\, ds$ depends only on the starting point & end point. (for all C)

$\Rightarrow \vec{F}$ is conservative

(Note: $\vec{F} = \vec{\nabla} f$ where $f(x,y) = x$)

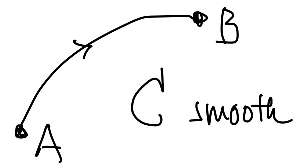
<u>Thm 8</u> (<u>Fundamental Theorem of Line Integral</u>)

Let f be a C^1 function on an open set $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, and $\vec{F} = \vec{\nabla} f$ be the gradient vector field of f. Then for any piecewise smooth oriented curve C on Ω with starting point A and end point B,

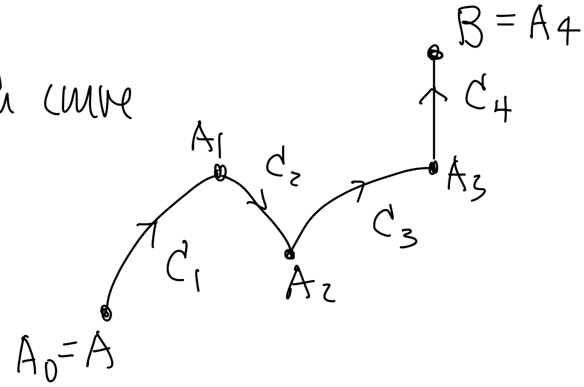
$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds = f(B) - f(A)}$$

Pf: Part 1 Assume C is a smooth curve parametrized by

$$\vec{r}(t), \quad a \le t \le b$$



Then
$$\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r}$$
$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$
$$= \int_a^b \vec{\nabla} f(\vec{r}(t)) \cdot \vec{r}'(t)\, dt$$
$$= \int_a^b \frac{d}{dt} f(\vec{r}(t))\, dt$$
$$= f(\vec{r}(b)) - f(\vec{r}(a)) \quad \text{(Fundamental Thm of Calculus in 1-var.)}$$
$$= f(B) - f(A)$$



Part 2 For a general piecewise smooth curve

$$C = C_1 \cup C_2 \cup \cdots \cup C_k$$

($= C_1 + C_2 + \cdots + C_k$ in order to indicate that they are joining end-by-end and the orientation C_i, $i = 1, \cdots, k$ are correct wrt the orientation of C)

where C_i is smooth going from A_{i-1} to A_i.

(then $A_0 = A$, $A_k = B$)

Then part 1 implies

$$\int_C \vec{F} \cdot \hat{T}\, ds = \int_{\sum_{i=1}^{k} C_i} \vec{F} \cdot \hat{T}\, ds$$

$$= \sum_{i=1}^{k} \int_{C_i'} \vec{F} \cdot \hat{T}\, ds \quad \text{(by def. 9')}$$

$$= \sum_{i=1}^{k} \left[f(A_i) - f(A_{i-1}) \right] \quad \text{(by part 1)}$$

$$= f(A_k) - f(A_0)$$

$$= f(B) - f(A) \quad \divideontimes$$

Is the converse of Thm 8 correct? Yes (under a further condition on the domain Ω)

Thm 9 Let $\Omega \subset \mathbb{R}^n$, $n = 2$ or 3, be open and connected. $\vec{F}$ is a continuous vector field on Ω. Then the following are equivalent.

(a) $\exists$ a C^1 function $f: \Omega \to \mathbb{R}$ such that

$$\vec{F} = \vec{\nabla} f$$

(b) $\oint_{C} \vec{F} \cdot d\vec{r} = 0$ along any closed curve C on Ω.

(c) $\vec{F}$ is conservative.

Remarks: (1) The function f in (a) of Thm 9 is called the potential function of $\vec{F}$. It is unique up to an additive constant:

$$\vec{\nabla}(f+c)=\vec{F}, \quad \forall \text{ const. } c.$$

(2) $\vec{F}=M\hat{i}+N\hat{j}+L\hat{k}=\vec{\nabla}f \Leftrightarrow Mdx+Ndy+Ldz=df$

(Same for 2-dim)

In this case, $Mdx+Ndy+Ldz$ (or $Mdx+Ndy$ in dim. 2) is called an exact differential form.

Pf: "(a) $\Rightarrow$ (b)"

If f is C^1 and $\vec{F}=\vec{\nabla}f$ and

$\vec{r}:[a,b]\to\Omega$ parametrizes the closed curve C.

then $\vec{r}(a)=\vec{r}(b) \overset{\text{denote}}{=\!=} A$

Fundamental Thm of Line Integral $\Rightarrow$

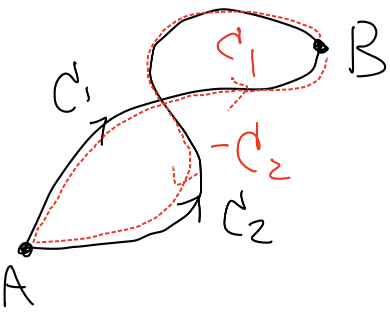
$$\oint_C \vec{F}\cdot d\vec{r} = f(\vec{r}(b))-f(\vec{r}(a)) = f(A)-f(A)=0$$

"(b) $\Rightarrow$ (c)" Suppose C_1 & C_2 are oriented curves with starting point A and end point B.

Then $C_1 - C_2$ (i.e. $C_1 \cup (-C_2)$)

is an oriented closed curve

(with starting point $= A =$ endpoint)



Then by part (b)

$$0 = \oint_{C_1 - C_2} \vec{F} \cdot d\vec{r} = \oint_{C_1} \vec{F} \cdot d\vec{r} + \oint_{(-C_2)} \vec{F} \cdot d\vec{r}$$

$$= \oint_{C_1} \vec{F} \cdot d\vec{r} - \oint_{C_2} \vec{F} \cdot d\vec{r}$$

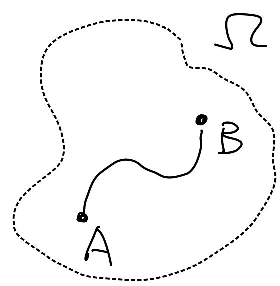
Since C_1 & C_2 are arbitrary, $\vec{F}$ is conservative.

"(c) $\Rightarrow$ (a)" (it requires us to solve a system of PDE.)

Assume $n=2$ for simplicity (other dimensions are similar)

Let $\vec{F} = M\hat{i} + N\hat{j}$ be conservative

Fix a point $A \in \Omega$



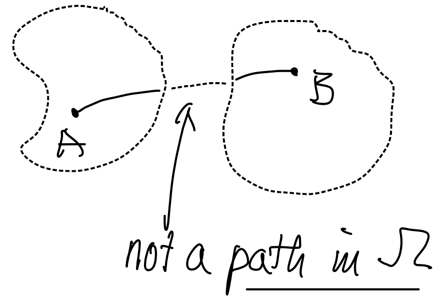
Then for any point $B \in \Omega$, define

$$f(B) = \int_A^B \vec{F} \cdot \hat{T}\, ds = \underset{\substack{\uparrow \\ (\vec{F} \text{ is conservative})}}{\underline{\text{common value}}} \text{ of } \int_C \vec{F} \cdot \hat{T}\, ds \text{ for any } C \text{ from } A \text{ to } B.$$

Since $\vec{F}$ is conservative, $f(B)$ is well-defined.

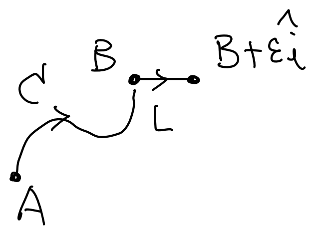
We've also used the assumption that Ω is <u>connected</u>.

Otherwise there is no path from A to B if A, B belong to different connected components.



<u>Claim</u> $\vec{F} = \vec{\nabla} f$

<u>Pf of claim</u>: $\frac{\partial f}{\partial x}(B) = \lim_{\varepsilon \to 0} \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon}$



Horizontal (straight) line segment from B to $B+\varepsilon\hat{i}$ with $|\varepsilon|$ sufficiently small such that $C+L \subset \Omega$ (open),

where C is an (arbitrary) oriented curve from A to B.

Then $f(B+\varepsilon\hat{i}) = \int_A^{B+\varepsilon\hat{i}} \vec{F}\cdot d\vec{r}$

$$= \int_{C+L} \vec{F}\cdot d\vec{r} = \int_C \vec{F}\cdot d\vec{r} + \int_L \vec{F}\cdot d\vec{r}$$

$$= f(B) + \int_L \vec{F}\cdot d\vec{r}$$

$$\therefore \quad \frac{f(B+\varepsilon\hat{i}) - f(B)}{\varepsilon} = \frac{1}{\varepsilon}\int_L \vec{F}\cdot d\vec{r} = \frac{1}{\varepsilon}\int_0^{\varepsilon} (M\hat{i} + N\hat{j})\cdot d\vec{r}$$

Since L can be parametrized by $(x+t, y)$, $0 \le t \le \varepsilon$

if $B = (x, y)$, we have

$$\frac{f(B+\varepsilon \hat{i}) - f(B)}{\varepsilon} = \frac{1}{\varepsilon} \int_0^{\varepsilon} \left[M(x+t, y)\hat{i} + N(x+t, y)\hat{j} \right] \cdot (\hat{i}\, dt)$$

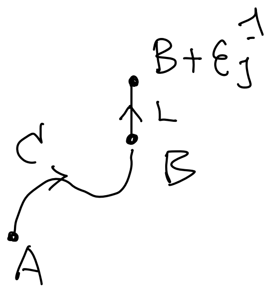
$$= \frac{1}{\varepsilon} \int_0^{\varepsilon} M(x+t, y)\, dt$$

$$\Rightarrow \quad \frac{\partial f}{\partial x}(B) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^{\varepsilon} M(x+t, y)\, dt$$

$$= M(x, y)$$ (by MVF & M is cts)
(or Fundamental Thm of Calculus)

Similarly $\frac{\partial f}{\partial y}(B) = N(x, y)$

by considering



(vertical line segment)

So $\vec{\nabla} f = \vec{F}$

Since $\vec{F}$ is cts, $\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$ are cts.

$\therefore$ f is C^1 ※

Corollary (to Thm 9)

Let $\vec{F}$ be conservative and C^1

"n = 3" If $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$ (on $\Omega \subset \mathbb{R}^3$) connected open

then
$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial N}{\partial x} \\[2mm] \dfrac{\partial N}{\partial z} = \dfrac{\partial L}{\partial y} \\[2mm] \dfrac{\partial L}{\partial x} = \dfrac{\partial M}{\partial z} \end{cases}$$

"n = 2" If $\vec{F} = M\hat{i} + N\hat{j}$ (on $\Omega \subset \mathbb{R}^2$) connected open

then $$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

Pf: $\vec{F}$ conservative $\overset{\text{Thm 9}}{\Longrightarrow}$ $\vec{F} = \vec{\nabla} f$ for some function f

i.e. $$\frac{\partial f}{\partial x}\hat{i} + \frac{\partial f}{\partial y}\hat{j} + \frac{\partial f}{\partial z}\hat{k} = M\hat{i} + N\hat{j} + L\hat{k}$$

$\vec{F} \in C^1 \Rightarrow f \in C^2$

Hence mixed derivatives thm (Clairaut's Thm)

$$\begin{cases} \dfrac{\partial M}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial N}{\partial x} \\[3mm] \dfrac{\partial N}{\partial z} = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial y}\right) = \dfrac{\partial}{\partial y}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial L}{\partial y} \\[3mm] \dfrac{\partial L}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{\partial f}{\partial z}\right) = \dfrac{\partial}{\partial z}\left(\dfrac{\partial f}{\partial x}\right) = \dfrac{\partial M}{\partial z} \end{cases}$$

("n = 2" is included) ※