Physics

(1) $\vec{F}$ = Force field

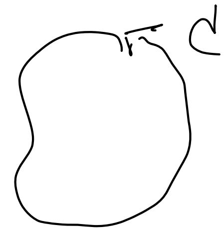
C = oriented curve

then $\boxed{W = \int_C \vec{F} \cdot \hat{T}\, ds}$

is the work done in moving an object along C.

(2) $\vec{F}$ = velocity vector field of fluid

C = oriented curve



Then $\boxed{\text{Flow} = \int_C \vec{F} \cdot \hat{T}\, ds}$

Flow along the curve C.

If C is "closed", the flow is also called a circulation.

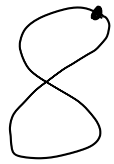
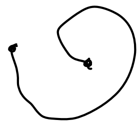
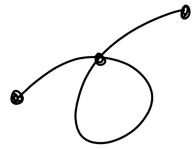
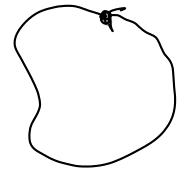
Def 13: A curve is said to be

(i) simple if it does not intersect with itself except possibly at end points.

(ii) closed if starting point = end point.

(iii) simple closed curve if it is both simple and closed.

Note:

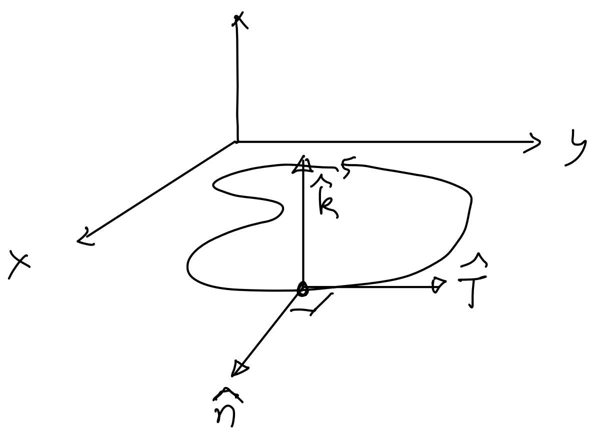
				
simple	NO	Yes	NO	Yes
closed	Yes	NO	NO	Yes

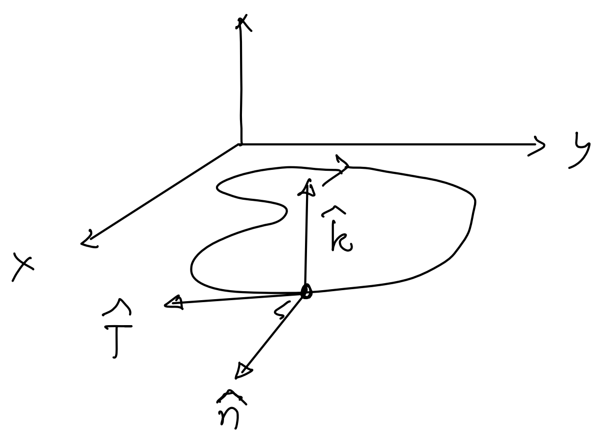
(3) $\vec{F}$ = velocity of fluid

C = oriented <u>plane</u> curve ($C \subset \mathbb{R}^2$) (simple, closed)

with parametrization $\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j}$

$\hat{n}$ = <u>outward-pointing unit normal (vector)</u> to the curve C





$$\boxed{\hat{n} = \hat{T} \times \hat{k}}$$

if C is of anti-clockwise orientation

$$\boxed{\hat{n} = -\hat{T} \times \hat{k}}$$

if C is of clockwise orientation.

Formula for $\hat{n}$ (wrt the parametrization $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$)

Recall $\hat{T} = \dfrac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \dfrac{x'(t)\hat{i} + y'(t)\hat{j}}{\|\vec{r}'(t)\|}$

(in arc-length parametrization: $\hat{T} = \dfrac{d\vec{r}}{ds} = \dfrac{dx}{ds}\hat{i} + \dfrac{dy}{ds}\hat{j}$)

<u>Anti-clockwise</u>:

$$\hat{n} = \hat{T} \times \hat{k} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \dfrac{x'}{\|\vec{r}'\|} & \dfrac{y'}{\|\vec{r}'\|} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$\Rightarrow \quad \hat{n} = \frac{y'(t)\hat{i} - x'(t)\hat{j}}{\|\vec{r}'(t)\|} \quad \left(\text{or } \hat{n} = \frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \right)$$

<u>Clockwise</u>: $\hat{n} = \dfrac{-y'(t)\hat{i} + x'(t)\hat{j}}{\|\vec{r}'(t)\|}$ $\left(\text{or } \hat{n} = -\dfrac{dy}{ds}\hat{i} + \dfrac{dx}{ds}\hat{j} \right)$

$$\boxed{\underline{\text{Flux}} \text{ of } \vec{F} \text{ across } C \overset{\text{def}}{=} \int_C \vec{F} \cdot \hat{n}\, ds}$$

(amount of fluid getting out of the closed curve C)

If $\vec{F} = M(x,y)\hat{i} + N(x,y)\hat{j}$

and $\vec{r}(t) = x(t)\hat{i} + y(t)\hat{j}$ is <u>anti-clockwise</u> parametrization of C (<u>closed curve</u>)

Then

Flux of $\vec{F}$ across C

$$= \ointctrclockwise_C (M\hat{i} + N\hat{j}) \cdot \left(\frac{dy}{ds}\hat{i} - \frac{dx}{ds}\hat{j} \right) ds$$

$$= \ointctrclockwise_C M\,dy - N\,dx$$

Remark:

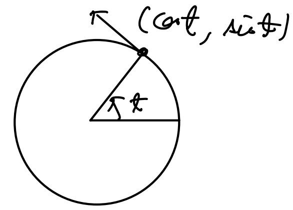
- $\ointctrclockwise$: curve is closed & in anti-clockwise direction
- $\ointclockwise$ = curve is closed & in clockwise direction

  (not a common notation)
- But in some books, only "$\oint$" is used, NO arrow, Then one needs to determine the orientation from the context.
- Convention: If no orientation is mentioned, "$\oint$" without arrow means anti-clockwise orientation (positive orientation)

eg 40: Let $\vec{F} = (x-y)\hat{i} + x\hat{j}$

$C: x^2+y^2 = 1$

Find the flow (anti-clockwisely) along C and flux across C.



Soln: Let $\vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j}$, $0 \le t \le 2\pi$

Note: correct orientation

Then flow $= \oint_C \vec{F}\cdot\hat{T}\,ds = \oint_C \vec{F}\cdot d\vec{r}$

$$= \int_0^{2\pi} \left[(\cos t - \sin t)\hat{i} + \cos t\,\hat{j}\right]\cdot\left[-\sin t\,\hat{i} + \cos t\,\hat{j}\right] dt$$

$$= \int_0^{2\pi} \left[-\sin t(\cos t - \sin t) + \cos^2 t\right] dt$$

$= \cdots = 2\pi$ (check!)

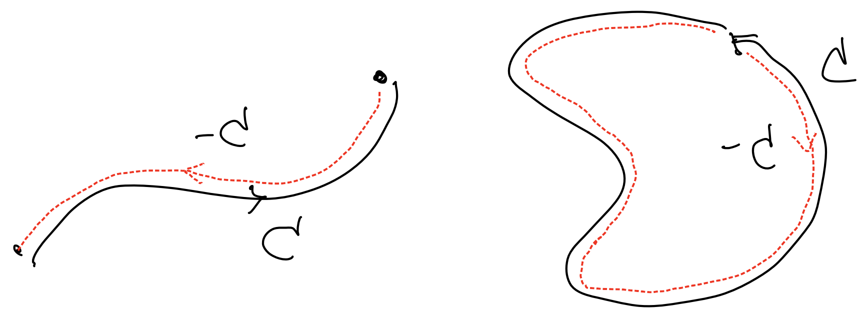
Flux $= \oint_C \vec{F}\cdot\hat{n}\,ds = \oint_C M\,dy - N\,dx$ $\quad \begin{pmatrix} M = x-y \\ N = x \end{pmatrix}$

$$= \int_0^{2\pi} (\cos t - \sin t)\,d(\sin t) - \cos t\,d(\cos t)$$

$$= \int_0^{2\pi} \left[(\cos t - \sin t)\cos t - \cos t(-\sin t)\right] dt$$

$= \cdots = \pi$ (check!)

Remark: If C is an oriented curve, then denote by "$-C$" the oriented curve with opposite orientation



- If f is a scalar function

$$\boxed{\int_C f\, ds = \int_{-C} f\, ds}$$

as "ds" is not oriented, just "length"

- If $\vec{F}$ is a vector field

flow $$\boxed{\int_C \vec{F}\cdot\hat{T}\, ds = -\int_{-C} \vec{F}\cdot\hat{T}\, ds}$$

this $\hat{T}$ is the "$\hat{T}$ for $-C$"

More precise formula:

$$\boxed{\int_C \vec{F}\cdot\hat{T}_C\, ds = -\int_{-C} \vec{F}\cdot\hat{T}_{-C}\, ds}$$

- But for flux $$\boxed{\oint_C \vec{F}\cdot\hat{n}\, ds = \oint_{-C} \vec{F}\cdot\hat{n}\, ds}$$ $\hat{n}$ always outward

Summary:

scalar f	$\int_C f\, ds$ indep. of orientation	ds have no direction
vector $\vec{F}$ flow	$\int_C \vec{F}\cdot\hat{T}\, ds$ depends on orientation	$\hat{T}$ depends on direction
flux	$\int_C \vec{F}\cdot\hat{n}\, ds$ indep. of orientation	$\hat{n}$ always outward