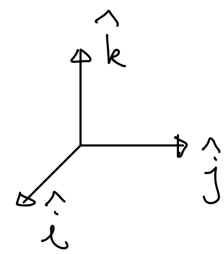
Vector Analysis

Notation: Usually in textbooks, vectors are denoted by boldface **i**, but hard to do it on screen, so my notation of vectors are:

$$\begin{cases} \text{general vectors}: & \vec{V},\ \vec{F},\ \vec{r},\ \vec{\nabla},\ \ldots \quad \text{(differential operator)} \\ \text{unit vectors}: & \hat{i},\ \hat{j},\ \hat{k},\ \hat{n},\ \hat{T},\ \ldots \end{cases}$$



and use $\|\vec{V}\|$ to denote the length of $\vec{V}$ to avoid confusion with absolute value $|x|$

Line integrals in $\mathbb{R}^3$ ($\mathbb{R}^n$)
(path integrals)

<u>Def 9</u>: The <u>line integral</u> of a <u>function</u> f on a <u>curve</u> (path, line) C with parametrization

$$\begin{array}{ccc} \vec{r}: [a,b] & \longrightarrow & \mathbb{R}^3 \\ \Psi & & \Psi \\ t & \longmapsto & (x(t), y(t), z(t)) \end{array}$$

(position vector)

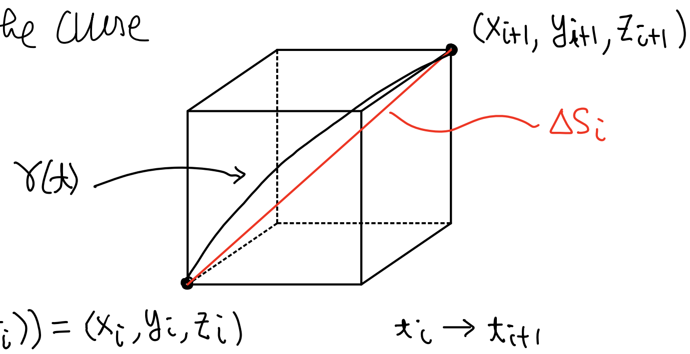
is

$$\int_C f(\vec{r})\, ds = \lim_{\|P\| \to 0} \sum_{i=1}^{n} f(\vec{r}(t_i)) \Delta s_i$$

where P is a partition of $[a,b]$, and

$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

i.e. ds = length element of the curve



<u>Remarks:</u>

(1) If $f \equiv 1$,

$$\boxed{\int_C ds = \text{arc-length of } C}$$

(2) The definition is well-defined, i.e. the RHS in the definition is independent of the parametrization $\vec{r}(t)$.

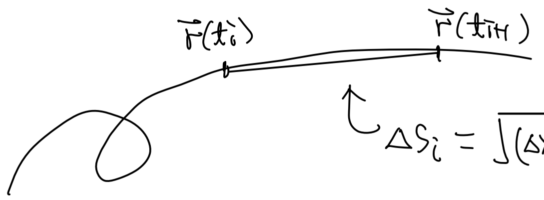
<u>Def 9'</u> (<u>Formula for line integral</u>)

Notations as in Def 9, then

$$\int_C f(\vec{r})\, ds = \int_a^b f(\vec{r}(t))\, \|\vec{r}\,'(t)\|\, dt$$

where $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$

Since



$$\Delta s_i = \sqrt{(\Delta x_i)^2 + (\Delta y_i)^2 + (\Delta z_i)^2}$$

$$= \sqrt{\left(\frac{\Delta x_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta y_i}{\Delta t_i}\right)^2 + \left(\frac{\Delta z_i}{\Delta t_i}\right)^2}\ \Delta t_i$$

$$\simeq \sqrt{x'(t_i)^2 + y'(t_i)^2 + z'(t_i)^2}\ \Delta t_i$$

$$= \|\vec{r}\,'(t_i)\|\ \Delta t_i$$

Remarks (1) "$ds = \|\vec{r}\,'(t)\|\,dt$" is usually referred as the arc-length element,

where $\vec{r}\,'(t) = (x'(t), y'(t), z'(t))$ and $\|\vec{r}\,'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

(2) Suppose the curve C is parametrized by a new parameter $\tilde{t}$

$$\begin{array}{ccc} t & \longleftrightarrow & \tilde{t} \\ \cap & & \cap \\ [a,b] & & [\tilde{a}, \tilde{b}] \end{array} \qquad \left(t \leftrightarrow \tilde{t} \text{ is increasing } \quad \frac{d\tilde{t}}{dt} > 0, \ \frac{dt}{d\tilde{t}} > 0 \right)$$

then $ds = \|\vec{r}\,'(t)\|\,dt = \left\| \dfrac{d\vec{r}}{dt}(t) \right\| dt$

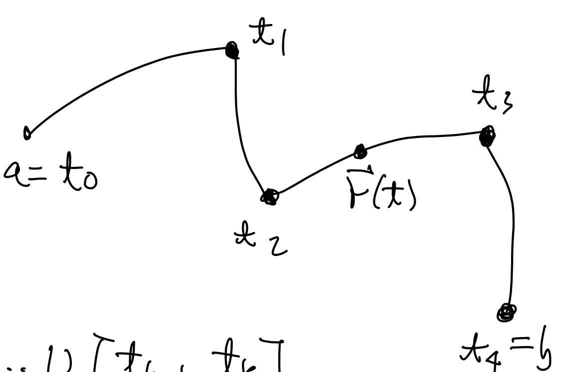
$$= \left\| \frac{d\vec{r}}{d\tilde{t}} \cdot \frac{d\tilde{t}}{dt} \right\| dt = \left\| \frac{d\vec{r}}{d\tilde{t}} \right\| \left| \frac{d\tilde{t}}{dt} \right| dt = \left\| \frac{d\vec{r}}{d\tilde{t}} \right\| d\tilde{t}$$

$\therefore$ ds and hence $\int_C f(\vec{r})\,ds$ is independent of the parametrization of C.

(3) If $\vec{r}(t)$ is only piecewise differentiable,

then the RHS of Def 9'

becomes a sum:



If $[a,b] = [t_0, t_1] \cup \cdots \cup [t_{i-1}, t_i] \cup \cdots \cup [t_{k-1}, t_k]$, where $t_0 = a$, $t_k = b$

non-differentiable point for $\vec{r}(t)$

such that $\vec{r}\big|_{[t_{i-1},\, t_i]}$ is differentiable, then

$$\boxed{\int_C f(\vec{r})\, ds = \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} f(\vec{r}(t)) \|\vec{r}'(t)\| dt}$$

eg 32 : $f(x,y,z) = x - 3y^2 + z$

C = (straight) line segment joining the origin and $(1,1,1)$

Find $\int_C f(x,y,z)\, ds$

Soln : Parametrize C by

$$\vec{r}(t) = (0,0,0) + t\left[(1,1,1) - (0,0,0)\right], \quad 0 \le t \le 1$$

$$= (t, t, t)$$

(i.e. $x(t) = t$, $y(t) = t$, $z(t) = t$)

$$\Rightarrow \quad \vec{r}'(t) = (1,1,1), \quad \forall t \in [0,1]$$

$$\Rightarrow \int_C f(\vec{r})\, ds = \int_0^1 \left[x(t) - 3y(t)^2 + z(t)\right] \|\vec{r}'(t)\|\, dt$$

$$= \int_0^1 (t - 3t^2 + t)\sqrt{3}\, dt = 0 \quad \text{(check!)} \quad \#$$

eg33 : Let C be a curve in $\mathbb{R}^2$ (plane curve) (i.e. $z(t) \equiv 0$)

and it has 2 parametrizations

$$\vec{r}_1(t) = (\cos t, \sin t), \qquad t \in [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]$$

$$\vec{r}_2(t) = (\sqrt{1-t^2}, -t), \qquad t \in [-1, 1]$$

Suppose $f(x,y) = x$. Find $\int_C f(x,y)\, ds$.

(We simply omit the z-variable, as C is a plane curve and f is indep. of z)

Soln: (1) $\vec{r}_1(t) = (\cos t, \sin t)$, $-\frac{\pi}{2} \le t \le \frac{\pi}{2}$

$$\int_C f(x,y)\, ds = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t \cdot \|(\cos t, \sin t)'\|\, dt$$

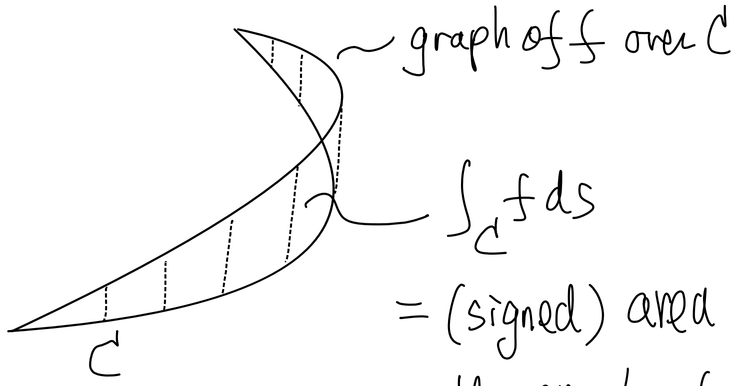
$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos t\, dt = 2 \qquad \text{(check!)}$$

(2) $\vec{r}_2(t) = (\sqrt{1-t^2}, -t)$, $-1 \le t \le 1$

$$\int_C f(x,y)\, ds = \int_{-1}^{1} \sqrt{1-t^2} \sqrt{\left(\frac{d}{dt}\sqrt{1-t^2}\right)^2 + \left(\frac{d}{dt}(-t)\right)^2}\, dt$$

$$= \cdots = \int_{-1}^{1} dt = 2 \qquad \text{(check!)}$$

This verifies the fact that the line integral is indep. of the parametrization. (Note: $\vec{r}_1(t)$ & $\vec{r}_2(t)$ are in opposite directions, see later discussion.)



$\int_C f\, ds$

= (signed) area under the graph of f over C. #

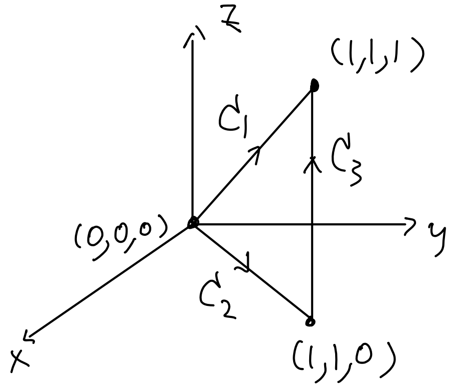
> **Prop 7:** If C is a piecewise smooth curve made by joining $C_1, C_2, \cdots C_n$ <u>end-to-end</u>, then
> $$\int_C f\, ds = \sum_{i=1}^{n} \int_{C_i} f\, ds$$

(Pf: Clear from the remark (3) of Def 9', but C_i can be piecewise differentiable in this Prop.)

<u>Remark</u>: "end-to-end" means "end point of C_{k-1} = initial point of C_k".

eg34 : let $f(x,y,z) = x - 3y^2 + z$ (again)

C_1, C_2, C_3 are (straight) line segments as in the figure



We already did $\int_{C_1} f\,ds = 0$ (eg32)

One can similarly calculate

$$\int_{C_2 \cup C_3} f\,ds = \int_{C_2} f\,ds + \int_{C_3} f\,ds$$

$$= -\frac{\sqrt{2}}{2} - \frac{3}{2} \qquad \text{(Ex!)}$$

(For instance, $\int_{C_3} f\,ds = \int_0^1 (1 - 3(1)^2 + t)\,dt$ (what parametrization?))

The observation is $\int_{C_1} f\,ds = 0 \neq \int_{C_2 \cup C_3} f\,ds$

even C_1 & $C_2 \cup C_3$ have the same beginning and end points!

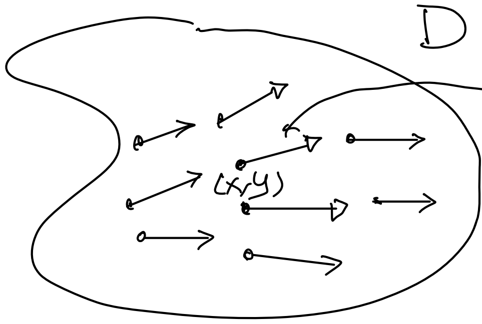
(Remark: different from 1-variable calculus)

Conclusion: Line integral of a function depends, not only on the end points, but also the path.

Vector Fields

Def 10: Let $D \subset \mathbb{R}^2$ or $\mathbb{R}^3$ be a region, then a <u>vector field</u> on D is a mapping $\vec{F}: D \to \mathbb{R}^2$ or $\mathbb{R}^3$ respectively

$D \subset \mathbb{R}^2$



$\vec{F}(x,y) \in \mathbb{R}^2$ considered as a 2-vector.

In <u>component form</u>:

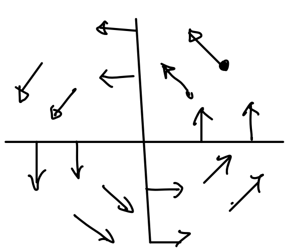
$$\mathbb{R}^2: \quad \vec{F}(x,y) = M(x,y)\hat{i} + N(x,y)\hat{j}$$

$$\mathbb{R}^3: \quad \vec{F}(x,y,z) = M(x,y,z)\hat{i} + N(x,y,z)\hat{j} + L(x,y,z)\hat{k}$$

where M, N, L are functions on D called the <u>components</u> of $\vec{F}$.

<u>eg 35</u> $\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ on $\mathbb{R}^2 \setminus \{(0,0)\}$

$= -\sin\theta\,\hat{i} + \cos\theta\,\hat{j}$ (in polar coordinates)



Properties of $\vec{F}$: (i) $\|\vec{F}(x,y)\| = 1$

(ii) $\vec{F} \perp \vec{r}(x,y) = x\hat{i} + y\hat{j} = r(\cos\theta\,\hat{i} + \sin\theta\,\hat{j})$

eg36 (Gradient vector field of a function)

(i) $f(x,y) = \frac{1}{2}(x^2+y^2)$

$$\vec{\nabla} f(x,y) \stackrel{\text{def}}{=\!=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) = (x,y) = x\hat{i} + y\hat{j} = \vec{r}(x,y)$$

position vector field.

(ii) $f(x,y,z) = x$

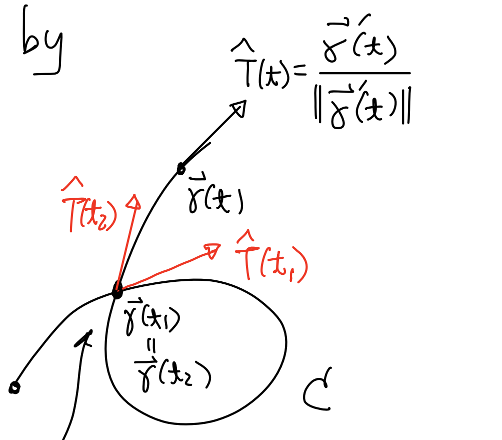
$$\vec{\nabla} f(x,y,z) \stackrel{\text{def}}{=\!=} \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right) \equiv (1,0,0) = \hat{i}$$

(a constant vector field)

eg37 (Vector field along a curve)

Let C be a curve in $\mathbb{R}^2$ parametrized by

$$\vec{\gamma} : [a,b] \to \mathbb{R}^2$$
$$\quad\ \ \Downarrow \qquad\quad \Downarrow$$
$$t \longmapsto (x(t), y(t)) = \vec{\gamma}(t)$$



Recall: $\hat{T} = \dfrac{\vec{\gamma}\,'(t)}{\|\vec{\gamma}\,'(t)\|}$

= unit tangent vector field along C (same point, but different vector)

Note: this $\hat{T}$ defined only on C (for a general curve),
but <u>not</u> outside C.

(vector field along a curve may not come from a vector field on a region.)

Remark: for eg37.

If we use $ds = \|\vec{r}'(t)\| dt$, then

$$\hat{T} = \frac{\vec{r}'(t)}{\|\vec{r}'(t)\|} = \frac{\frac{d\vec{r}}{dt}}{\frac{ds}{dt}} = \frac{d\vec{r}}{ds}$$ (by Chain rule)
(if s is a function of t)

where "arc-length s" is defined by
(up to an additive constant)

$$s(t) = \int_{t_0}^{t} \|\vec{r}'(t)\| dt,$$

A parametrization of a curve C by arc-length s is called <u>arc-length parametrization</u>:

$\vec{r}(s)$ = arc-length parametrization

$$\Rightarrow \left\| \frac{d\vec{r}}{ds}(s) \right\| = 1$$

<u>Def 11</u> A vector field is defined to be <u>continuous</u> / <u>differentiable</u> / <u>C^k</u> if the component functions are.

<u>eg38</u>:

$\vec{F}(x,y) = \vec{r}(x,y) = x\hat{i} + y\hat{j}$ is C^∞ (position vector)

$\vec{F}(x,y) = \dfrac{-y\hat{i} + x\hat{j}}{\sqrt{x^2+y^2}}$ is <u>not</u> continuous in $\mathbb{R}^2$
(but continuous in $\mathbb{R}^2 \setminus \{(0,0)\}$)

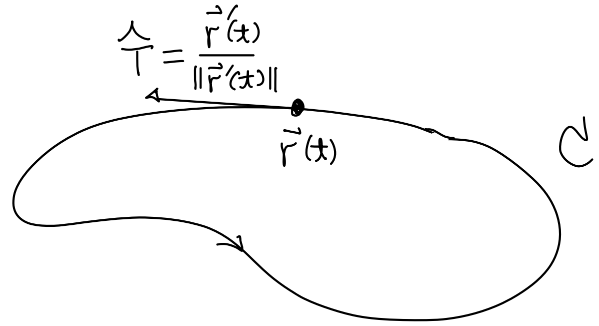
Line integral of vector field

<u>Def 12</u>: Let C be a curve with "<u>orientation</u>" given by a parametrization $\vec{r}(t)$ with $\vec{r}\,'(t) \neq 0$, $\forall t$. Define the <u>line integral of a vector field $\vec{F}$ along C</u> to be

$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds}$$

where $\hat{T} = \dfrac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|}$ is the unit tangent vector field along C.

(i.e. C is oriented in the <u>direction</u> of $\vec{r}\,'(t)$ or $\hat{T}$ at every point)



<u>Note</u>: If $\vec{r} : [a,b] \to \mathbb{R}^n$ ($n = 2$ or 3) then

$$\int_C \vec{F} \cdot \underbrace{\hat{T}\, ds} = \int_a^b \vec{F}(r(t)) \cdot \frac{\vec{r}\,'(t)}{\|\vec{r}\,'(t)\|} \|\vec{r}\,'(t)\|\, dt$$

$$= \int_a^b \vec{F}(\vec{r}(t)) \cdot \underbrace{\vec{r}\,'(t)\, dt}_{d\vec{r}}$$

(Abuse of notations:
$\vec{r}(t) = \vec{r}(x(t), y(t)) = x(t)\hat{i} + y(t)\hat{j}$
because of the position vector field
$\vec{r} = x\hat{i} + y\hat{j}$)

$\therefore$ naturally, we denote $\boxed{d\vec{r} = \hat{T}\, ds}$

and $$\boxed{\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}}$$

eg 38 : $\vec{F}(x,y,z) = z\hat{i} + xy\,\hat{j} - y^2\hat{k}$

$C : \vec{r}(t) = t^2\hat{i} + t\hat{j} + \sqrt{t}\,\hat{k}, \quad 0 \le t \le 1$

Soln $$d\vec{r} = \left(2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right)dt$$

& $$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r}$$

$$= \int_0^1 \left(\sqrt{t}\,\hat{i} + t^2\cdot t\,\hat{j} - t^2\hat{k}\right)\cdot\left(2t\,\hat{i} + \hat{j} + \frac{1}{2\sqrt{t}}\hat{k}\right)dt$$

$$= \int_0^1 \left(2t\sqrt{t} + t^3 - \frac{t^2}{2\sqrt{t}}\right)dt = \frac{17}{20} \quad \text{(check!)}$$

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In components form:

Line integral of $\vec{F} = M\hat{i} + N\hat{j}$ along

$$C : \vec{r}(t) = g(t)\,\hat{i} + h(t)\,\hat{j}$$

can be expressed as

$$\int_C \vec{F}\cdot\hat{T}\,ds = \int_C \vec{F}\cdot d\vec{r} = \int_a^b \left(\vec{F}\cdot\frac{d\vec{r}}{dt}\right)dt$$

$$= \int_a^b (Mg' + Nh')\,dt$$

(more explicitly: $\int_a^b [M(g(t), h(t))\, g'(t) + N(g(t), h(t))\, h'(t)]\, dt$)

Note that, $\begin{cases} x = g(t) \\ y = h(t) \end{cases}$

$$\Rightarrow \begin{cases} dx = g'(t)\, dt \\ dy = h'(t)\, dt \end{cases}$$

$$\therefore \boxed{\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b M\, dx + N\, dy}$$

Similarly, for 3-dim

$$\boxed{\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_a^b M\, dx + N\, dy + L\, dz}$$

(for $\vec{F} = M\hat{i} + N\hat{j} + L\hat{k}$)

<u>Another way to justify the notation</u>:

$\vec{r} = (x, y, z)$ the position vector

$\Rightarrow$ $\boxed{d\vec{r} = (dx, dy, dz)}$ (natural notation)

Then $\int_C \vec{F} \cdot \hat{T}\, ds = \int_C \vec{F} \cdot d\vec{r} = \int_C (M, N, L) \cdot (dx, dy, dz)$

$$= \int_C M\, dx + N\, dy + L\, dz.$$

eg 39 : Evaluate $I = \int_C -y\,dx + z\,dy + 2x\,dz$

where $C : \vec{r}(t) = \cos t\,\hat{i} + \sin t\,\hat{j} + t\,\hat{k} \quad (0 \le t \le 2\pi)$

$$= (\cos t, \sin t, t)$$

Soln $I = \int_0^{2\pi} (-\sin t\, d\cos t + t\, d\sin t + 2\cos t\, dt)$

$$= \int_0^{2\pi} (-\sin t \cdot (-\sin t) + t \cdot \cos t + 2\cos t)\,dt$$

$$= \cdots = \pi \quad (\text{Check!})$$

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$(d\vec{r} = (-\sin t, \cos t, 1)\,dt \quad \& \quad \vec{r}'(t) = (-\sin t, \cos t, 1))$