eg31 Let $D = \{(x,y,z) \in \mathbb{R}^3 : |x|+|y|+|z| \le 1\}$

Evaluate $\iiint_D (x+y+z)^4 \, dV$.

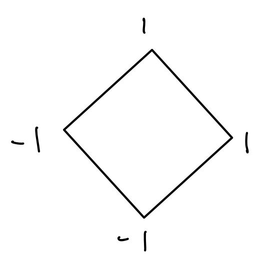
Remarks: (1) One may use symmetry $(x,y,z) \Leftrightarrow (-x,-y,-z)$ to reduce half, but not to the 1st octant using reflections, since for instance,

$$x+y+z \longleftrightarrow x+y-z \quad \text{under } (x,y,z) \leftrightarrow (x,y,-z),$$

$\therefore (x+y+z)^4$ is not symmetric in all reflections with respect to the coordinates lines

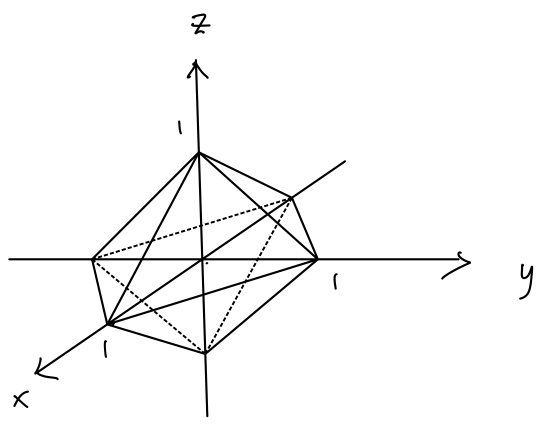
(2) Of course, one may calculate directly without changing variables (Ex!)

Soln If $z=0$, then $|x|+|y| \le 1$



boundary lines are $\begin{cases} x+y = \pm 1 \\ x-y = \pm 1 \end{cases}$

Hence



Boundary surfaces (planes) are

$$\begin{cases} x+y+z = \pm 1 \\ x+y-z = \pm 1 \\ x-y-z = \pm 1 \\ x-y+z = \pm 1 \end{cases}$$

$$(*)\quad \text{Let}\ \begin{cases} u = x+y+z \\ v = x+y-z \\ w = x-y-z \end{cases} \quad \text{then} \quad \begin{cases} -1 \leq u \leq 1 \\ -1 \leq v \leq 1 \\ -1 \leq w \leq 1 \end{cases}$$

remaining pair

$x-y+z = \pm 1$ become $u-v+w = \pm 1$

Change of variables formula $\Rightarrow$

$$\iiint_D (x+y+z)^4 dV = \iiint_{\substack{-1\leq u,v,w\leq 1 \\ -1 \leq u-v+w \leq 1}} u^4 \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dv\, dw\, du$$

By solving $(*)$, $\begin{cases} x = \frac{1}{2}(u+w) \\ y = \frac{1}{2}(v-w) \\ z = \frac{1}{2}(u-v) \end{cases}$ (check!)

$$\left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| = \left| \det \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & 0 \end{pmatrix} \right| = \left| -\frac{1}{4} \right| = \frac{1}{4} \quad \text{(check!)}$$

$$\text{Hence} \quad \iiint_D (x+y+z)^4 dV = \iiint_{\substack{-1\leq u,v,w\leq 1 \\ -1 \leq u-v+w \leq 1}} \frac{u^4}{4}\, dv\, dw\, du$$

$$= A - B - C$$

$$\text{where} \quad A = \iiint_{-1\leq u,v,w\leq 1} \frac{u^4}{4}\, dv\, dw\, du$$

$$B = \iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \ge 1}} \frac{u^4}{4} \, dv \, dw \, du$$

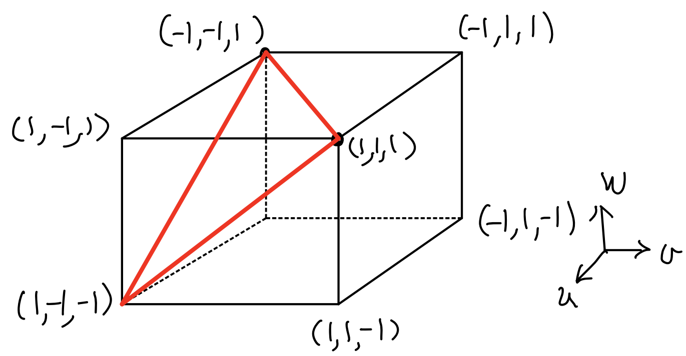
$$C = \iiint_{\substack{-1 \le u,v,w \le 1 \\ u-v+w \le -1}} \frac{u^4}{4} \, dv \, dw \, du$$

(Observation: $B = C$ by symmetry $(u,v,w) \leftrightarrow (-u,-v,-w)$)

It is clear that

$$A = \iiint_{-1 \le u,v,w \le 1} \frac{u^4}{4} \, dv \, dw \, du = \frac{2}{5} \quad \text{(easy ex!)}$$

To handle B (& C)



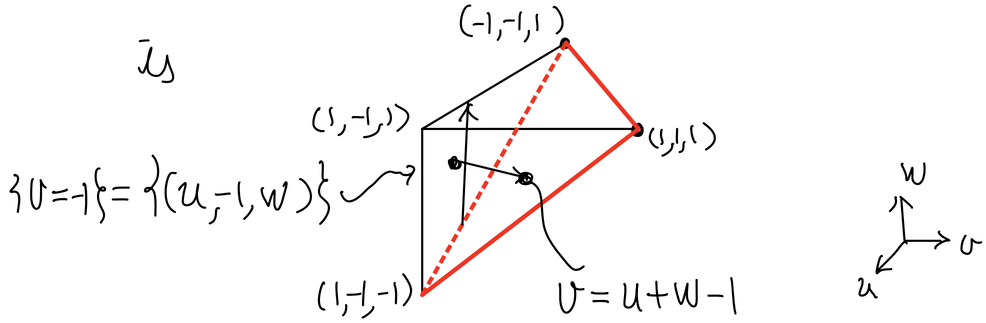
	$u-v+w$
$(1,-1,1)$	3
$(1,1,1)$	1
$(-1,-1,1)$	1
$(1,-1,-1)$	1

Hence the 3 points $(1,1,1)$, $(-1,-1,1)$, $(1,-1,-1)$ are on the boundary plane $u-v+w=1$ of B.

Since plane determined by 3 points, so $u-v+w=1$ is the plane passing through these 3 points.

So the solid for integration B, $\left\{ \begin{array}{l} -1 \leq u, v, w \leq 1 \\ u - v + w \geq 1 \end{array} \right\}$

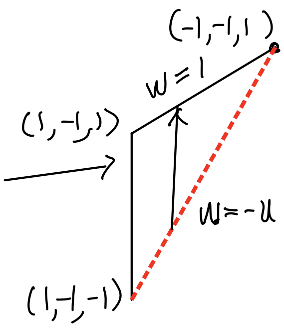
is



which of special type

$$B = \iint_R \left[\int_{-1}^{u+w-1} \frac{u^4}{4}\, dv \right] dw\, du$$

where R



$$= \int_{-1}^{1} \left[\int_{-u}^{1} \left[\int_{-1}^{u+w-1} \frac{u^4}{4}\, dv \right] dw \right] du$$

$$= \cdots = \frac{3}{35} \qquad \text{(check!)}$$

Symmetry $\Rightarrow$ $C = B = \frac{3}{35}$

(The solid for the integration C is determined by the 4 points $(-1,1,-1)$, $(-1,-1,-1)$, $(1,1,-1)$ & $(-1,1,1)$)

Finally $\iiint_D (x+y+z)^4\, dV = A - B - C = \frac{2}{5} - 2 \cdot \frac{3}{35} = \frac{8}{35}$

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