Change of Variable Formula

Review of 1-variable

In Riemann sum

$$\int_a^b f(x)\,dx = \int_{[a,b]} f(x)\,\underbrace{dx}_{} \quad (\sim |\Delta x| = \text{length of subinterval} > 0)$$

↑ as set (we don't care about the direction)

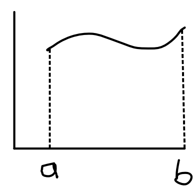
If $a > b$

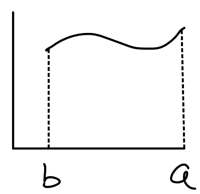
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx = -\int_{[b,a]} f(x)\,dx$$

([b, a] & [a, b] represent the same set)

In summary

$$\int_a^b f(x)\,dx = \begin{cases} \int_{[a,b]} f(x)\,dx, & \text{if } a \le b \\ -\int_{[a,b]} f(x)\,dx, & \text{if } a \ge b \end{cases}$$





([a, b] = [b, a] as set = $\{x : x \text{ between } a \,\&\, b\}$)

Change of variable in 1-variable

$$\int_a^b f(x)\,dx = \int_c^d \left[f(x(u))\,\frac{dx}{du} \right] du$$

where $c = u(a)$, $d = u(b)$.

If $\frac{dx}{du} > 0$, then $d = u(b) > u(a) = c$

$$\therefore \int_a^b f(x)\,dx = \int_{[c,d]} \left[f(x(u))\frac{dx}{du}\right] du$$

$$= \int_{[c,d]} f(x(u)) \left|\frac{dx}{du}\right| du$$

If $\frac{dx}{du} < 0$, then $d = u(b) < u(a) = c$

$$\therefore \int_a^b f(x)\,dx = \int_c^d \left[f(x(u))\frac{dx}{du}\right] du = -\int_{[d,c]} f(x(u))\frac{dx}{du}\, du$$

$$= \int_{[d,c]} f(x(u)) \left|\frac{dx}{du}\right| du$$

Hence (in Riemann sum)

$$\int_{[a,b]} f(x)\,dx = \int_{[c,d]} f(x) \left|\frac{dx}{du}\right| du$$

interpreted as a set without direction

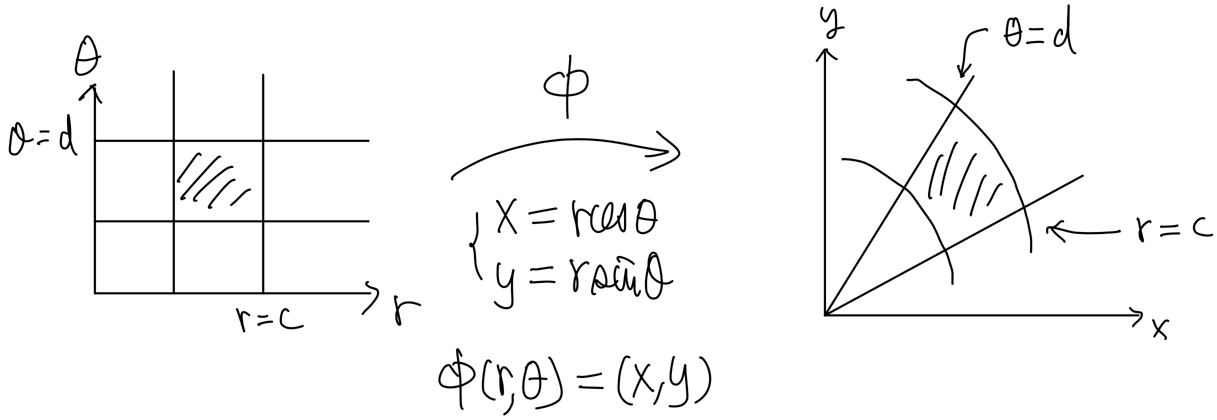
(i.e. $\{u : u$ between c & d (inclusive)$\}$)

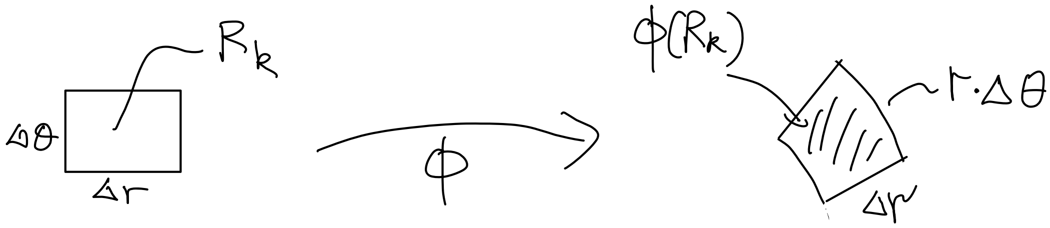
& $\left|\frac{dx}{du}\right| \sim \frac{|\Delta x|}{|\Delta u|}$ ratio of <u>lengths</u> (of the coordinates)

1-dim'l

Generalization to multiple integrals

<u>eg</u>: Polar coordinates: $\iint\limits_{(x,y)} f(x,y)\,dx\,dy = \iint\limits_{(r,\theta)} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$





$\text{Area}(R_k) \cong \Delta r \Delta\theta$ $\qquad$ $\text{Area}(\phi(R_k)) \cong r\Delta r\Delta\theta$

Hence

$$\frac{\text{Area}(\phi(R_k))}{\text{Area}(R_k)} \to r \quad \text{as "} R_k \to \text{point"}$$

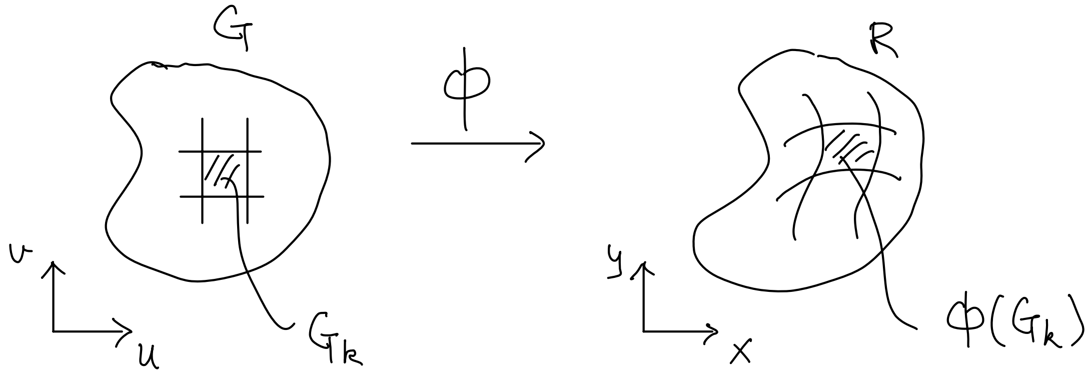
(ratio of <u>areas</u>, <u>always ≥ 0</u>)

2-dim'l

General change of coordinate formula in $\mathbb{R}^2$

Suppose $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$ is denoted by $\phi(u,v) = (x,y)$,

$$\phi : \underset{\text{uv-plane}}{G} \longrightarrow \underset{\text{xy-plane}}{R}$$



Idea: We need to find

$$\frac{\text{Area}(\phi(G_k))}{\text{Area}(G_k)} \longrightarrow \,? \quad \text{as "} G_k \to \text{point"}$$

Assume ϕ is a diffeomorphism: 1-1, onto & $\phi, \phi^{-1} \in C^1$.

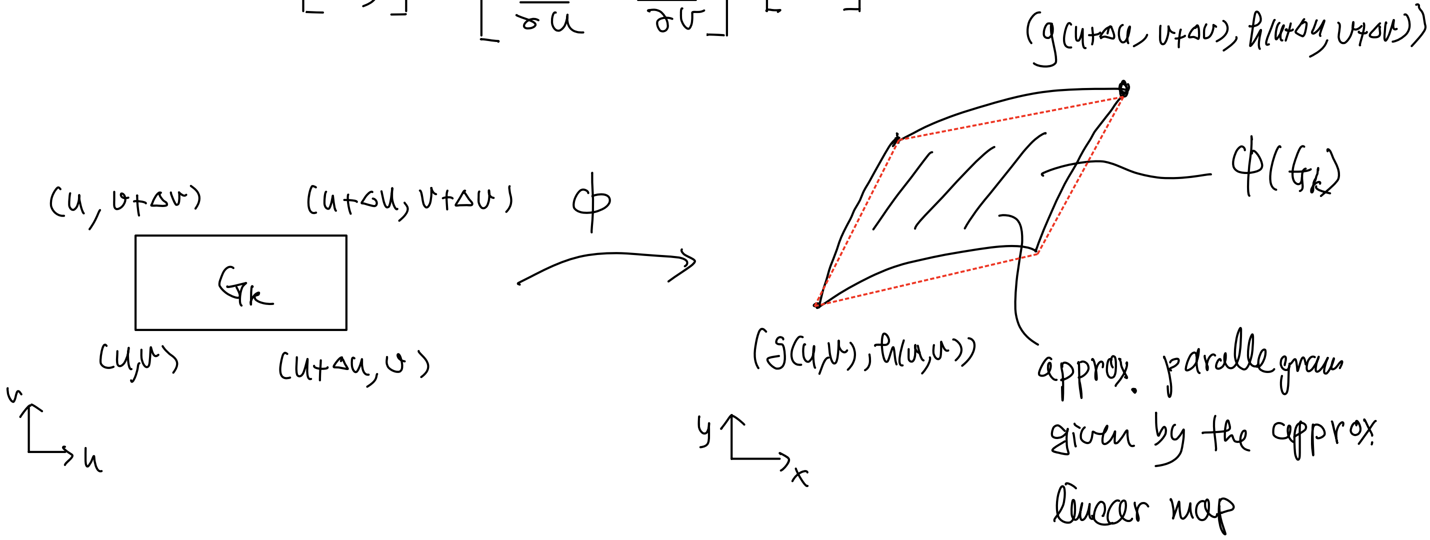
ϕ is $C^1 \Rightarrow$

$$\begin{cases} g(u+\Delta u, v+\Delta v) = g(u,v) + \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ h(u+\Delta u, v+\Delta v) = h(u,v) + \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

$$\Rightarrow \begin{cases} \Delta x = \Delta g = g(u+\Delta u, v+\Delta v) - g(u,v) = \dfrac{\partial g}{\partial u}\Delta u + \dfrac{\partial g}{\partial v}\Delta v + \cdots \\ \Delta y = \Delta h = h(u+\Delta u, v+\Delta v) - h(u,v) = \dfrac{\partial h}{\partial u}\Delta u + \dfrac{\partial h}{\partial v}\Delta v + \cdots \end{cases}$$

In matrix form

$$\begin{bmatrix} \Delta x \\ \Delta y \end{bmatrix} = \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} \begin{bmatrix} \Delta u \\ \Delta v \end{bmatrix} + \cdots$$



(By linear algebra)

$$\frac{dA_{(x,y)}}{dA_{(u,v)}} \cong \frac{\text{Area}(\Phi(G_k))}{\text{Area}(G_k)} \cong \left| \det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} \end{bmatrix} \right|$$

$$\cong$$

$$\left(\frac{\text{"}\Delta x \Delta y\text{"}}{\text{"}\Delta u \Delta v\text{"}} \right) \qquad = \left| \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} \right|$$

<u>Def 7</u>: Define the <u>Jacobian</u> $J(u,v)$ of the "coordinates transformation" $\begin{cases} x = g(u,v) \\ y = h(u,v) \end{cases}$

by

$$J(u,v) \overset{\text{notation}}{=\!=} \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix}$$

With this notation, we should have the formula

$$\boxed{\begin{aligned}\iint\limits_{\substack{R\\ ((x,y)\in)}} f(x,y)\,dx\,dy &= \iint\limits_{\substack{G\\ ((u,v)\in)}} f(g(u,v), h(u,v))\left|\det\begin{bmatrix}\frac{\partial g}{\partial u} & \frac{\partial g}{\partial v}\\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v}\end{bmatrix}\right| du\,dv\\ &= \iint\limits_G f(x(u,v), y(u,v))\,|J(u,v)|\,du\,dv\\ &= \iint\limits_G f(x(u,v), y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv\end{aligned}}$$

<u>eg 28</u>: $\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$ $\quad\big((u,v) = (r,\theta)\big)$

$$\Rightarrow \quad J(r,\theta) = \frac{\partial(x,y)}{\partial(r,\theta)} = \det\begin{bmatrix}\frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta}\\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta}\end{bmatrix} = r \quad \text{(check!)}$$

and

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_G f(r\cos\theta, r\sin\theta)\left|\frac{\partial(x,y)}{\partial(r,\theta)}\right| dr\,d\theta$$

$$= \iint\limits_G f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta$$

(same formula as before.)

Thm 6: Suppose $\phi : \begin{pmatrix} u \\ v \end{pmatrix} \mapsto \begin{pmatrix} x \\ y \end{pmatrix}$ is a diffeomorphism (1-1, onto, s.t. ϕ and $\phi^{-1} \in C^1$) mapping a region G (closed and bounded) in the uv-plane onto a region R (closed and bounded) in the xy-plane (except possibly on the boundary).

Suppose $f(x,y)$ is __continuous__ on R, then

$$\iint_R f(x,y)\,dx\,dy = \iint_G f\circ\phi(u,v) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| du\,dv$$

__Notes__: (i) $f\circ\phi(u,v) = f(x(u,v), y(u,v))$

(ii) ϕ is a diffeomorphism $\Rightarrow \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \neq 0$.

__Triple integrals__ ("substitutions" in triple integrals)

$$\phi(u,v,w) = (x,y,z) : G^{\subset \mathbb{R}^3 \ni (u,v,w)} \longrightarrow D^{\subset \mathbb{R}^3 \ni (x,y,z)}$$

with
$$\begin{cases} x = g(u,v,w) \\ y = h(u,v,w) \\ z = k(u,v,w) \end{cases}$$
1-1, onto, cont. differentiable and __inverse__ also cont. differentiable.

__Def 8__ __Jacobian__ (__determinant__) of transformation in $\mathbb{R}^3$

$$J(u,v,w) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{bmatrix} \left(= \det \begin{bmatrix} \frac{\partial g}{\partial u} & \frac{\partial g}{\partial v} & \frac{\partial g}{\partial w} \\ \frac{\partial h}{\partial u} & \frac{\partial h}{\partial v} & \frac{\partial h}{\partial w} \\ \frac{\partial k}{\partial u} & \frac{\partial k}{\partial v} & \frac{\partial k}{\partial w} \end{bmatrix} \right)$$

Note: Chain rule $\Rightarrow$

$$\begin{cases} \text{2-dim} & \dfrac{\partial(x,y)}{\partial(u,v)} \cdot \dfrac{\partial(u,v)}{\partial(s,t)} = \dfrac{\partial(x,y)}{\partial(s,t)} \\ \text{3-dim} & \dfrac{\partial(x,y,z)}{\partial(u,v,w)} \cdot \dfrac{\partial(u,v,w)}{\partial(s,t,r)} = \dfrac{\partial(x,y,z)}{\partial(s,t,r)} \end{cases}$$

(Ex!)

$$\Rightarrow \begin{cases} \text{2-dim} & \dfrac{\partial(u,v)}{\partial(x,y)} = \dfrac{1}{\dfrac{\partial(x,y)}{\partial(u,v)}} \\ \text{3-dim} & \dfrac{\partial(u,v,w)}{\partial(x,y,z)} = \dfrac{1}{\dfrac{\partial(x,y,z)}{\partial(u,v,w)}} \end{cases}$$

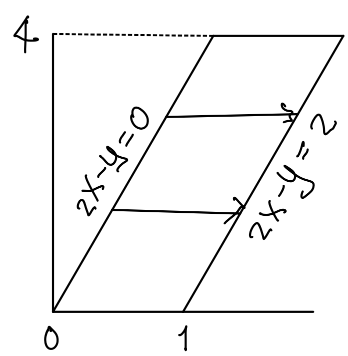
(Ex!)

Thm 7: Under similar conditions of Thm 6

$$\iiint_D F(x,y,z)\,dx\,dy\,dz = \iiint_G F\circ\phi(u,v,w)\,|J(u,v,w)|\,du\,dv\,dw$$

$$= \iiint_G F\big(g(u,v,w), h(u,v,w), k(u,v,w)\big)\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

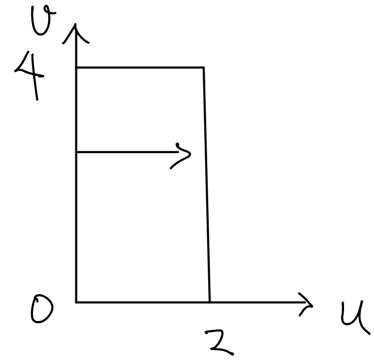
eg 29 $\displaystyle\int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\,dx\,dy$



Soln lower limit $x=\frac{y}{2} \Leftrightarrow 2x-y=0$

upper limit $x=\frac{y}{2}+1 \Leftrightarrow 2x-y=2$

Define $\begin{cases} u = 2x - y \\ v = y \end{cases}$



Then

$$\begin{cases} x = \frac{1}{2}u + \frac{1}{2}v \\ y = v \end{cases}$$

$$\begin{cases} 2x - y = 0 \leftrightarrow u = 0 \\ 2x - y = 2 \leftrightarrow u = 2 \end{cases}$$

$$\begin{cases} y = 0 \\ y = 4 \end{cases} \leftrightarrow \begin{matrix} v = 0 \\ v = 4 \end{matrix}$$

$$J(u,v) = \frac{\partial(x,y)}{\partial(u,v)} = \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{bmatrix} = \det \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \frac{1}{2}$$

$$\therefore \int_0^4 \int_{\frac{y}{2}}^{\frac{y}{2}+1} \frac{2x-y}{2}\, dx\, dy = \int_0^4 \int_0^2 \frac{u}{2} \cdot \left|\frac{1}{2}\right| du\, dv = 2 \quad (\text{check!})$$

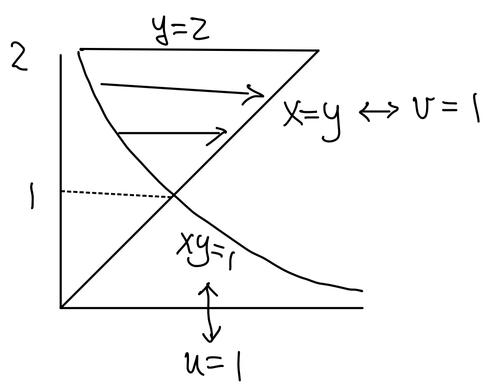
#

<u>eg30</u> $I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$

<u>Soln</u>:

Let $\begin{cases} u = \sqrt{xy} \\ v = \sqrt{\frac{y}{x}} \end{cases}$

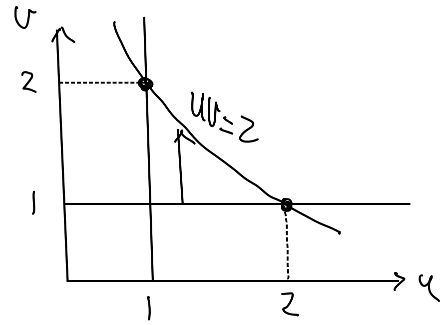
Domain of integration



Express x, y in terms of u, v

$$\begin{cases} x = \frac{u}{v} \\ y = uv \end{cases}$$

Then $\begin{cases} x=y \longleftrightarrow v=1 \\ x=\frac{1}{y} \longleftrightarrow u=2 \\ y=2 \longleftrightarrow uv=2 \end{cases}$



$$\frac{\partial(x,y)}{\partial(u,v)} = \det\begin{pmatrix} \frac{1}{v} & -\frac{u}{v^2} \\ v & u \end{pmatrix} = \frac{2u}{v}$$

$$I = \int_1^2 \int_{\frac{1}{y}}^{y} \sqrt{\frac{y}{x}}\, e^{\sqrt{xy}}\, dx\, dy$$

$$= \int_1^2 \int_1^{\frac{2}{u}} v e^u \cdot \left|\frac{2u}{v}\right| dv\, du \quad \left(\text{or } \int_1^2 \int_1^{\frac{2}{v}} v e^u \cdot \left|\frac{2u}{v}\right| du\, dv\right)$$

$$= \int_1^2 \int_1^{\frac{2}{u}} 2ue^u\, dv\, du$$

$$= \int_1^2 \left(2ue^u \int_1^{\frac{2}{u}} dv\right) du$$

$$= \int_1^2 2ue^u\left(\frac{2}{u}-1\right) du$$

$$= 2e(e-2) \qquad (\text{check!}) \qquad ※$$

<u>eg18 (revisit)</u> Volume of Ellipsiod

$$D = \left\{ \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c>0)$$

$$\mathrm{Vol}(D) = 8\int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

Soln Change of variables

$$\begin{cases} u = \frac{x}{a} \\ v = \frac{y}{b} \\ w = \frac{z}{c} \end{cases} \leftrightarrow \quad \begin{array}{l} x = au \\ y = bv \\ z = cw \end{array}$$

"New" domain in (u,v,w)-space

is $G = \{ (u,v,w) : u^2 + v^2 + w^2 \leq 1 \}$

which is the unit ball in (u,v,w)-space.

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det \begin{bmatrix} a & & \\ & b & \\ & & c \end{bmatrix} = abc$$

$$\therefore \ Vol(D) = 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} dz\, dy\, dx$$

$$= 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} \left| \frac{\partial(x,y,z)}{\partial(u,v,w)} \right| dw\, dv\, du$$

$$= abc \cdot 8 \int_0^1 \int_0^{\sqrt{1-u^2}} \int_0^{\sqrt{1-u^2-v^2}} dw\, dv\, du$$

$$= abc \cdot Vol(\text{solid unit ball in } (u,v,w)\text{-space})$$

$$\left(= abc \int_0^{2\pi} \int_0^{\pi} \int_0^1 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta \right)$$

$$= \frac{4\pi}{3} abc$$

※