eg25

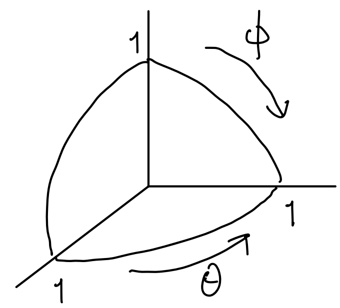
$$f(x,y,z) = \begin{cases} \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}}, & \text{if } (x,y,z) \neq (0,0,0) \\ 0, & \text{if } (x,y,z) = (0,0,0) \end{cases}$$

(In fact, f is continuous, but it is sufficient to know f is continuous except at the origin $(0,0,0)$)

Let D = unit ball centered at origin intersecting with the 1st octant

Find the <u>average</u> of f over D.

<u>Soln:</u> D can be represented in spherical coordinates:



$$\begin{cases} 0 \leq \rho \leq 1 \\ 0 \leq \phi \leq \frac{\pi}{2} \\ 0 \leq \theta \leq \frac{\pi}{2} \end{cases}$$

And $f(x,y,z) = \dfrac{x^2+y^2}{\sqrt{x^2+y^2+z^2}} = \dfrac{\rho^2 \sin^2\phi}{\rho}$ $\quad ((x,y,z) \neq (0,0,0))$

$= \rho \sin^2\phi \quad$ ($\therefore f \to 0$ as $\rho \to 0 \Rightarrow f$ is cts at 0)

Hence $$\iiint_D f(x,y,z)\,dV = \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \underbrace{(\rho\sin^2\phi)}_{\text{function}} \cdot \underbrace{\rho^2 \sin\phi \, d\rho\, d\phi\, d\theta}_{\text{volume element}}$$

$$= \int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1 \rho^3 \sin^3\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{\pi}{2}\left(\int_0^{\frac{\pi}{2}} \sin^3\phi \, d\phi\right)\left(\int_0^1 \rho^3 d\rho\right)$$

$$= \frac{\pi}{12} \quad \text{(check!)}$$

$$\text{Vol}(D) = \frac{1}{8}\text{Vol(unit ball)} = \frac{1}{8}\cdot\frac{4\pi}{3} = \frac{\pi}{6}$$

$$\Rightarrow \text{Average of } f \text{ over } D = \frac{1}{\text{Vol}(D)}\iiint_D f(x,y,z)\,dV$$

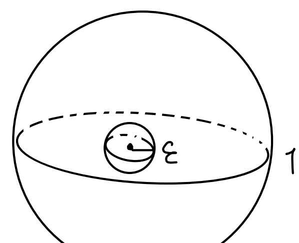
$$= \frac{1}{2} \qquad \#$$

<u>eg 26</u>: (Improper integrals)

Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2} = \dfrac{1}{\rho^2}$

$g(x,y,z) = \dfrac{1}{(\sqrt{x^2+y^2+z^2})^3} = \dfrac{1}{\rho^3}$ (both unbounded as $\rho \to 0$)

over unit ball $B = \{(\rho,\theta,\phi) : 0 \le \rho \le 1\}$



(i) Does $\displaystyle\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} f(x,y,z)\,dV$ exist?

where $B_\varepsilon = \{(\rho,\phi,\theta) : 0 \le \rho \le \varepsilon\}$

(ii) Does $\displaystyle\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV$ exist?

Answer:

(i) $$\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} f(x,y,z)\,dV = \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^2}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \lim_{\varepsilon\to 0} 2\pi\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_\varepsilon^1 d\rho\right)$$

$$= \lim_{\varepsilon\to 0} 4\pi(1-\varepsilon) = 4\pi$$

(ii) $$\lim_{\varepsilon\to 0}\iiint_{B\setminus B_\varepsilon} g(x,y,z)\,dV = \lim_{\varepsilon\to 0}\int_0^{2\pi}\int_0^{\pi}\int_\varepsilon^1 \frac{1}{\rho^3}\cdot\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

$$= \lim_{\varepsilon\to 0} 4\pi\left(\int_\varepsilon^1 \frac{d\rho}{\rho}\right) = \lim_{\varepsilon\to 0} 4\pi\ln\frac{1}{\varepsilon}$$

Doesn't exist ※

Terminology:
- $f = \frac{1}{\rho^2}$ is said to be "integrable" over B (in the sense of improper integral)
- $g = \frac{1}{\rho^3}$ is said to be "non integrable" over B

Question: determine all $\beta > 0$ such that

$f = \frac{1}{\rho^\beta}$ is "integrable" over $B \subset \mathbb{R}^3$

Similar question in $\mathbb{R}^2$: determine all $\beta > 0$ such that

$f = \frac{1}{r^\beta}$ is "integrable" in $\{r \le 1\} \subset \mathbb{R}^2$

(even in $\mathbb{R}^1$: $f = \frac{1}{|x|^\beta}$)

Application of Multiple Integrals (Thomas' Calculus §15.6)

In applications, we often use the following:

In 2-dim: Let R be a region in $\mathbb{R}^2$ with density $\delta(x,y)$

- First moment about y-axis: $M_y = \iint_R x\,\delta(x,y)\,dA$
- First moment about x-axis: $M_x = \iint_R y\,\delta(x,y)\,dA$
- Mass: $M = \iint_R \delta(x,y)\,dA$
- Center of Mass (Centroid)

$$(\bar{x},\bar{y}) = \left(\frac{M_y}{M}, \frac{M_x}{M}\right)$$

In 3-dim, D solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

- First moment:
  - about yz-plane, $M_{yz} = \iiint_D x\,\delta(x,y,z)\,dV$
  - about xz-plane, $M_{xz} = \iiint_D y\,\delta(x,y,z)\,dV$
  - about xy-plane, $M_{xy} = \iiint_D z\,\delta(x,y,z)\,dV$
- Mass: $M = \iiint_D \delta(x,y,z)\,dV$
- Center of Mass (Centroid) $(\bar{x},\bar{y},\bar{z}) = \left(\frac{M_{yz}}{M}, \frac{M_{xz}}{M}, \frac{M_{xy}}{M}\right)$

In 2-dim, R = region in $\mathbb{R}^2$ with density $\delta(x,y)$

Moments of inertia

- about x-axis : $I_x = \iint_R y^2 \delta(x,y)\,dA$
- about y-axis : $I_y = \iint_R x^2 \delta(x,y)\,dA$
- about line L : $I_L = \iint_R r(x,y)^2 \delta(x,y)\,dA$

where $r(x,y)$ = distance between (x,y) and L.

- about the origin : $I_0 = \iint_R (x^2+y^2)\delta(x,y)\,dA$

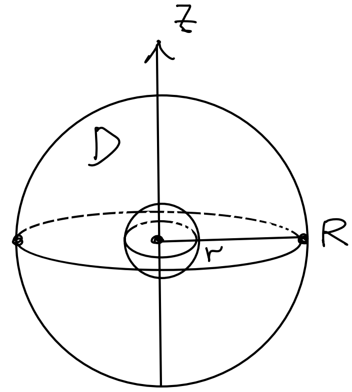
In 3-dim, D = solid region in $\mathbb{R}^3$ with density $\delta(x,y,z)$

Moments of Inertia

- around x-axis : $I_x = \iiint_D (y^2+z^2)\delta(x,y,z)\,dV$
- around y-axis : $I_y = \iiint_D (x^2+z^2)\delta(x,y,z)\,dV$
- around z-axis : $I_z = \iiint_D (x^2+y^2)\delta(x,y,z)\,dV$
- around Line L : $I_L = \iiint_D r(x,y,z)^2 \delta(x,y,z)\,dV$

where $r(x,y,z)$ = distance between (x,y,z) and L.

eg 27: Consider $D: r^2 \le x^2+y^2+z^2 \le R^2$
$(0<r<R)$



with density $\delta(x,y,z) \equiv \delta$
(constant density function, i.e. uniform mass)

Express I_z in term of m = Mass of D, r and R.

Soln: $$I_z \stackrel{\text{def}}{=\!=} \iiint_D (x^2+y^2)\,\delta(x,y,z)\,dV$$

$$= \delta \iiint_D (x^2+y^2)\,dV = \delta \int_0^{2\pi}\int_0^{\pi}\int_r^R (\rho\sin\phi)^2 \cdot \rho^2\sin\phi\, d\rho\, d\phi\, d\theta$$

$$= \delta \cdot 2\pi \left(\int_0^{\pi} \sin^3\phi\, d\phi\right)\left(\int_r^R \rho^4\, d\rho\right)$$

$$= \frac{8\pi}{15}(R^5-r^5)\,\delta \qquad \text{(check!)}$$

$$m = \text{Mass} = \iiint_D \delta(x,y,z)\,dV = \delta \iiint_D dV = \delta\, \text{Vol}(D)$$

$$= \delta \cdot \frac{4\pi}{3}(R^3-r^3)$$

$$\therefore \quad \boxed{I_z = \frac{2m}{5}\,\frac{R^5-r^5}{R^3-r^3}}$$

Observation: Two limiting cases:

(i) $r \to 0$, i.e. the whole solid ball $\quad \boxed{I_z = \frac{2m}{5}R^2}$

(ii) $r \to R$, i.e. a (hollow) sphere made of "infinitesimally" thin sheet:

$$I_z = \lim_{r \to R} \frac{2m}{5} \cdot \frac{R^5 - r^5}{R^3 - r^3} = \frac{2m}{5} \cdot \frac{5R^4}{3R^2} \quad \text{(check!)}$$

$$\therefore \quad \boxed{I_z = \frac{2m}{3} R^2}$$

Moment of inertia of the hollow sphere

> moment of inertia of the solid ball

(assuming the same (uniform) mass m) ⨳