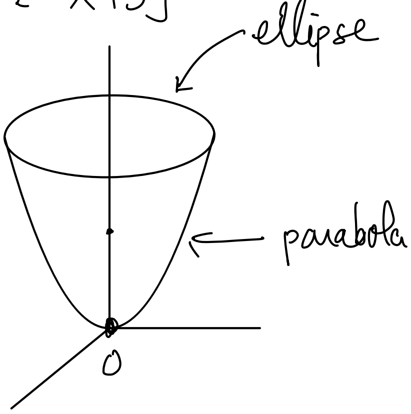
eg19: Find the volume of D enclosed by

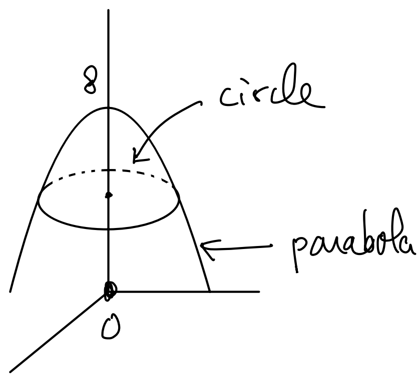
$$z = x^2 + 3y^2 \quad \text{and} \quad z = 8 - x^2 - y^2$$

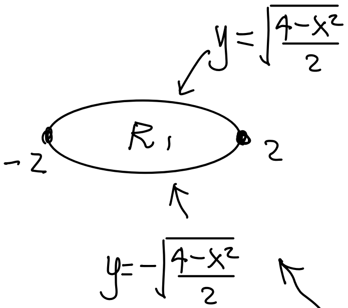
Soln:

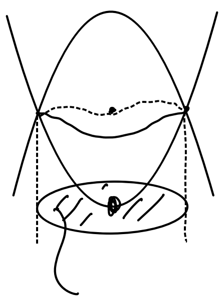
$z = x^2 + 3y^2$



$z = 8 - x^2 - y^2$







R_1
enclosed by the ellipse $x^2+2y^2=4$

This 2 surfaces intersect at the curve given

$$x^2 + 3y^2 = z = 8 - x^2 - y^2$$

with projection onto the xy-plane given by

$$x^2 + 2y^2 = 4$$

which is a ellipse.

$$\Rightarrow \quad D = \{ (x,y) \in R_1 = \{ x^2 + 2y^2 \leq 4 \},\ x^2 + 3y^2 \leq z \leq 8 - x^2 - y^2 \}$$

$$= \{ -2 \leq x \leq 2,\ -\sqrt{\frac{4-x^2}{2}} \leq y \leq \sqrt{\frac{4-x^2}{2}},\ x^2 + 3y^2 \leq z \leq 8 - x^2 - y^2 \}$$

$$\text{Vol}(D) = \int_{-2}^{2} \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} \int_{x^2+3y^2}^{8-x^2-y^2} 1 \, dz \, dy \, dx$$

$$= \int_{-2}^{2} \frac{4\sqrt{2}}{3}(4-x^2)^{3/2}\,dx \qquad \text{(check!)}$$

$$= 8\pi\sqrt{2} \qquad \text{(check!)}$$

[For those interested, the intersection (space) curve in parameter form is

$$x = 2\cos t,\ y = \sqrt{2}\sin t,\ z = 4 + 2\sin^2 t \qquad (0 \le t \le 2\pi)$$]

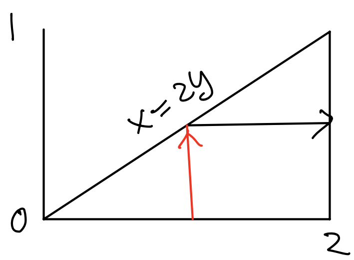
<u>eg 20</u> Evaluate $\displaystyle\int_0^4 \int_0^1 \int_{2y}^{2} \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$

<u>Soln</u>

$$\int_0^4 \int_0^1 \int_{2y}^{2} \frac{4\cos(x^2)}{2\sqrt{z}}\,dx\,dy\,dz$$

$$= \int_0^4 \frac{2}{\sqrt{z}}\left(\int_0^1 \int_{2y}^{2} \cos(x^2)\,dx\,dy\right)dz$$

$$= \underbrace{\left(\int_0^1 \int_{2y}^{2} \cos(x^2)\,dx\,dy\right)}_{\text{think of this as a double integral over the region}}\left(\int_0^4 \frac{2}{\sqrt{z}}\,dz\right)$$



By Fubini's

$$\int_0^4 \int_0^1 \int_{2y}^2 \frac{4\cos(x^2)}{2\sqrt{z}}\, dx\, dy\, dz$$

$$= \left(\int_0^2 \int_0^{\frac{x}{2}} \cos(x^2)\, dy\, dx\right)\left(\int_0^4 \frac{2}{\sqrt{z}}\, dz\right)$$

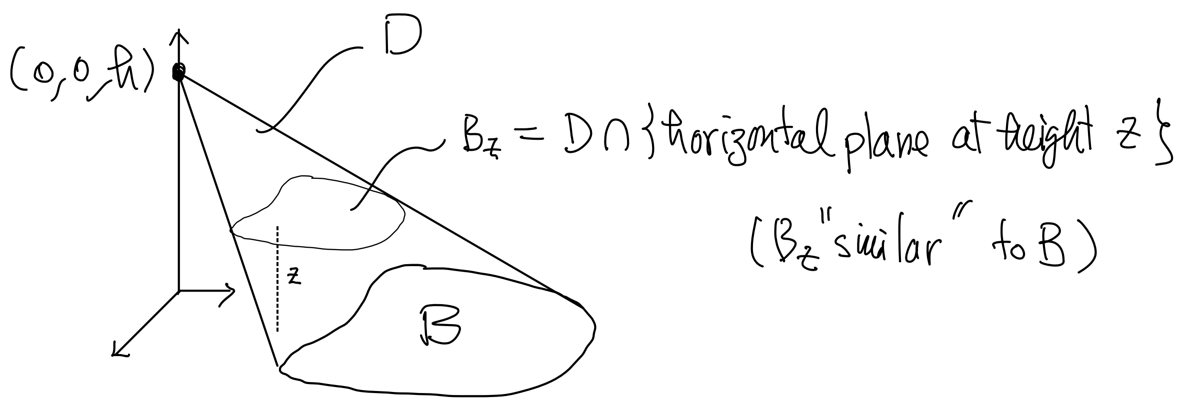
$$= \left(\int_0^2 \cos(x^2)\left(\int_0^{\frac{x}{2}} dy\right) dx\right)\left(\int_0^4 \frac{2}{\sqrt{z}}\, dz\right)$$

$$= \left(\int_0^2 \frac{x}{2}\cos(x^2)\, dx\right)\left(\int_0^4 \frac{2}{\sqrt{z}}\, dz\right)$$

$= 2\sin 4$ (check!) (using integration-by-parts) #

eg21 Let B (base) be a "nice" subset of $\mathbb{R}^2$.

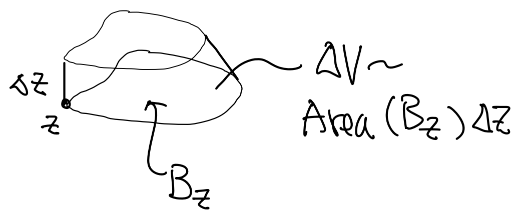
Let D = cone in $\mathbb{R}^3$ with base B on xy-plane and vertex $(0,0,h)$ $(h>0)$



How to find the volume of D?

Answer: By the concept of Riemann sum

$$\mathrm{Vol}(D) = \int_0^h \mathrm{Area}(B_z)\, dz$$



ratio of heights: $\frac{h-z}{h} = 1 - \frac{z}{h}$

ratio of areas: $\frac{\mathrm{Area}(B_z)}{\mathrm{Area}(B)} = \left(1 - \frac{z}{h}\right)^2$ by "similarity"

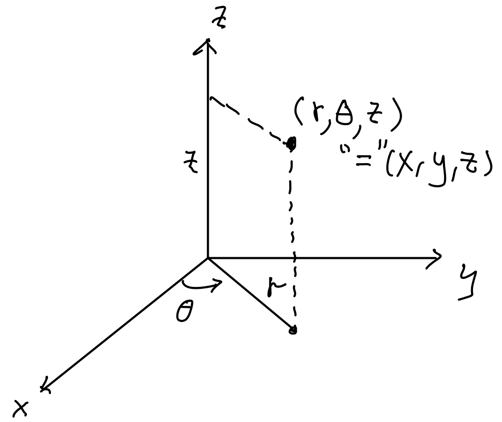
$$\Rightarrow \quad \mathrm{Vol}(D) = \int_0^h \left(1-\frac{z}{h}\right)^2 \mathrm{Area}(B)\, dz$$

$$= \mathrm{Area}(B) \int_0^h \left(1 - \frac{z}{h}\right)^2 dz$$

$$= \frac{h}{3}\mathrm{Area}(B) \quad \#$$

(check!)

Cylindrical Coordinates in $\mathbb{R}^3$



- (r,θ) = polar coordinates for the xy-plane $(r \geq 0)$

- z = rectangular vertical coordinate

Then a point $P=(x,y,z)$ can be represented by (r,θ,z), where

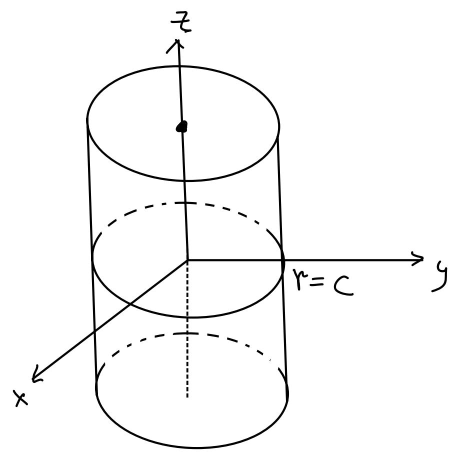
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z = z \end{cases}$$

And (r,θ,z) is called the <u>cylindrical coordinates</u> for $\mathbb{R}^3$.

<u>Remark 1</u>: (Let c be a constant)

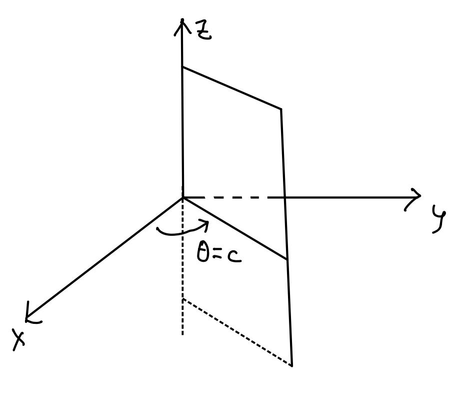
- $r = c \quad (c>0)$

  describes a <u>cylinder</u>



- $\theta = c \quad (0 \leq c \leq 2\pi)$

  describes a vertical half-plane



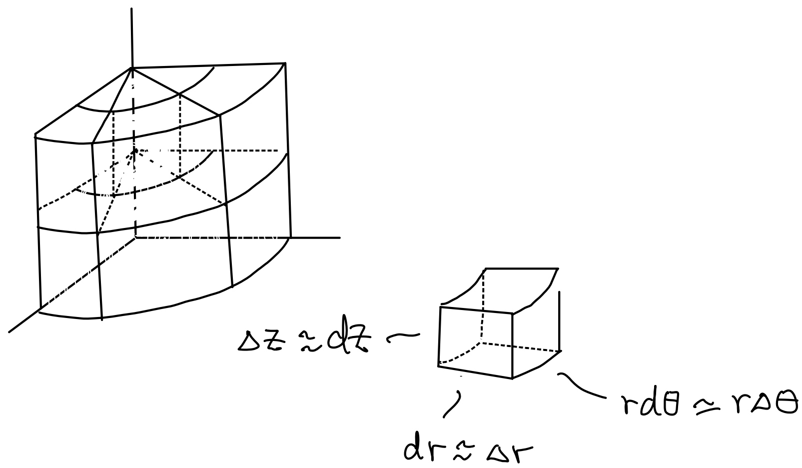
- $z = c$ describes a horizontal plane (as in rectangular coordinates)

Remark 2: We can define cylindrical coordinates in other directions:

e.g. $\begin{cases} x = x \\ y = r\cos\theta \\ z = r\sin\theta \end{cases}$ (Ex: draw the cylinder $r = c$)

Volume element

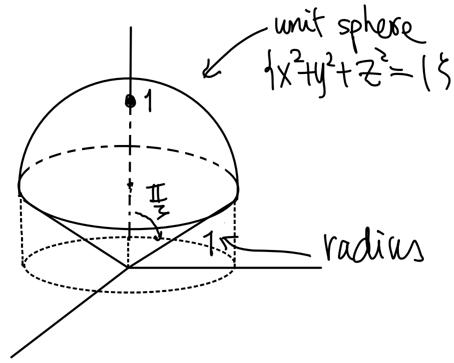
$$dV = \underbrace{dx\,dy}_{\updownarrow}\,\underbrace{dz}_{\updownarrow}$$
$$= r\,dr\,d\theta \cdot dz$$



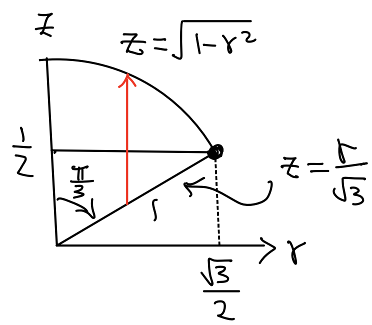
(order of the integration can be changed)

eg 22 (see also eg 24)

Find the volume of the Ice-cream cone I given as in the figure.



Soln: θ fixed



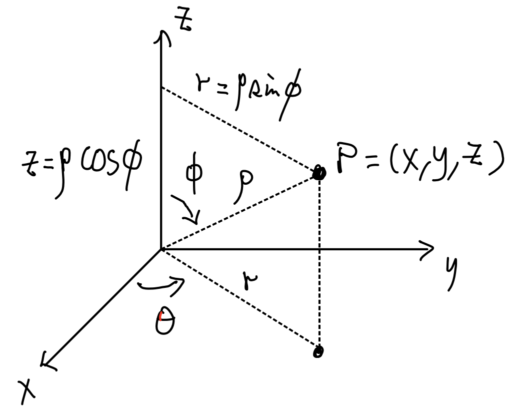
Fubini's $\Rightarrow$

$$\mathrm{Vol}(D) = \int_0^{2\pi}\left(\int_0^{\frac{\sqrt{3}}{2}}\left(\int_{\frac{r}{\sqrt{3}}}^{\sqrt{1-r^2}} dz\right) r\,dr\right) d\theta$$

$$= 2\pi \int_0^{\frac{\sqrt{3}}{2}} \left(\sqrt{1-r^2} - \frac{r}{\sqrt{3}}\right) r\,dr = \frac{\pi}{3} \text{ (check!)}$$

Spherical coordinates in $\mathbb{R}^3$

(ρ, ϕ, θ) where



- ρ = distance from the origin
  $(\rho \geq 0)$

- ϕ = angle from the <u>positive</u>
  z-axis to $\overline{OP}$ $\quad (0 \leq \phi \leq \pi)$

- θ = angle from cylindrical coordinate
  $(0 \leq \theta \leq 2\pi)$

<u>Remark</u>: If (r, θ, z) is the cylindrical coordinates of the point P, then
$$\begin{cases} r = \rho\sin\phi \\ z = \rho\cos\phi \end{cases}$$
(ρ, ϕ can be regarded as polar coordinates of the (z, r) coordinates)

In particular $\quad z^2 + r^2 = \rho^2$.

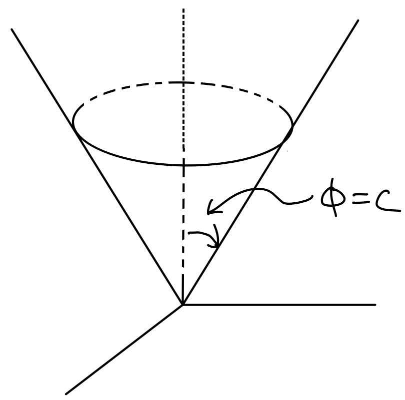
Then

rectangular		cylindrical		spherical
x	$=$	$r\cos\theta$	$=$	$\rho\sin\phi\cos\theta$
y	$=$	$r\sin\theta$	$=$	$\rho\sin\phi\sin\theta$
z	$=$	z	$=$	$\rho\cos\phi$

<u>Remark</u>: If c is a constant, then

- $\rho = c$ $(c > 0)$ describes a <u>sphere of radius c</u>
- $\theta = c$ describes a vertical half-plane.
- $\phi = c$ describes

$$= \begin{cases} +\text{ve } z\text{-axis}, & \text{if } c = 0 \\ -\text{ve } z\text{-axis}, & \text{if } c = \pi \\ xy\text{-plane}, & \text{if } c = \frac{\pi}{2} \\ \text{cone}, & \text{otherwise} \end{cases}$$

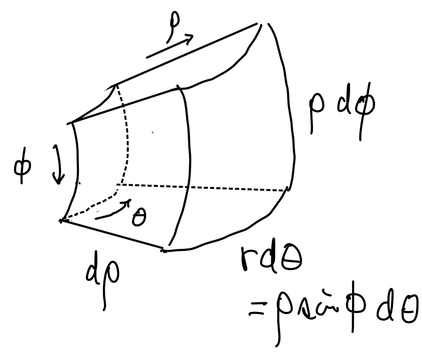


(upward $0 < c < \frac{\pi}{2}$
downward $\frac{\pi}{2} < c < \pi$)

<u>Volume element</u>

$$dV = dx\,dy\,dz = r\,dr\,d\theta\,dz$$

$$= (\rho \sin\phi)(\rho\,d\rho\,d\phi)\,d\theta$$



i.e. $\boxed{dV = \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta}$

<u>eg23</u> Convert the following into spherical coordinates

(1) $x^2 + y^2 + (z-1)^2 = 1$ (sphere)

(2) $z = -\sqrt{x^2+y^2}$ (cone)

Soln: (1) Sub
$$\begin{cases} x = \rho\sin\phi\cos\theta \\ y = \rho\sin\phi\sin\theta \\ z = \rho\cos\phi \end{cases} \quad \text{—— (*)}$$

into $x^2+y^2+(z-1)^2 = 1$

$$\Leftrightarrow \quad \rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta + (\rho\cos\phi - 1)^2 = 1$$

$$\Leftrightarrow \quad \rho^2\sin^2\phi + \rho^2\cos^2\phi - 2\rho\cos\phi = 0$$

$$\Leftrightarrow \quad \rho^2 = 2\rho\cos\phi \qquad (\rho \geq 0)$$

$$\Leftrightarrow \quad \rho = 2\cos\phi$$

(2) Sub. (*) into $z = -\sqrt{x^2+y^2}$ $(= -r)$

$$\Rightarrow \quad \rho\cos\phi = -\rho\sin\phi \qquad (\rho \geq 0)$$

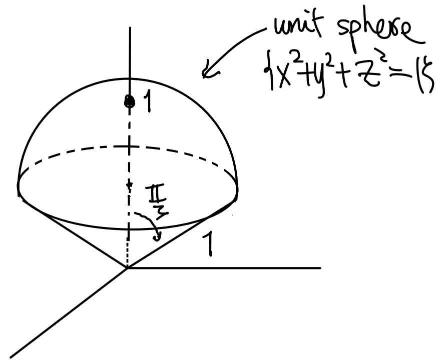
($\rho = 0$ is the point $(0,0,0)$)

& $\rho \neq 0 \Rightarrow \cos\phi = -\sin\phi \qquad (0 \leq \phi \leq \pi)$

$$\Rightarrow \quad \phi = \frac{3\pi}{4} \qquad \#$$

eg24 (see eg22)

Volume of ice-cream cone I again, in spherical coordinates



Soln: The ice-cream cone I is given by

$$\left\{ 0 \leq \rho \leq 1,\ 0 \leq \phi \leq \frac{\pi}{3},\ 0 \leq \theta \leq 2\pi \right\}$$

$$\mathrm{Vol}(I) = \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \underbrace{\rho^2 \sin\phi}\, d\rho\, d\phi\, d\theta$$

Don't miss this.

$$= \left(\int_0^{2\pi} d\theta\right)\left(\int_0^{\frac{\pi}{3}} \sin\phi\, d\phi\right)\left(\int_0^1 \rho^2 d\rho\right) = \frac{\pi}{3} \quad (\text{check!})$$

$\#$