Triple Integrals

Def 5 Let $f(x,y,z)$ be a function defined on a (closed and bounded) rectangular box

$$B = [a,b] \times [c,d] \times [r,s]$$

Then the triple integral of f over the box B is

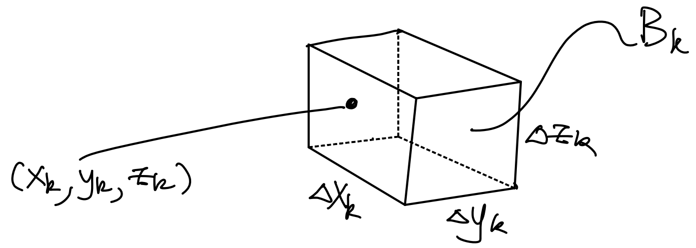
$$\iiint_B f(x,y,z)\,dV = \lim_{\|P\| \to 0} \sum_k f(x_k, y_k, z_k)\,\Delta V_k$$

if it exists.

where (i) $P = P_1 \times P_2 \times P_3$ is a subdivision of B into sub-rectangular boxes by partitions P_1, P_2 & P_3 of $[a,b]$, $[c,d]$, and $[r,s]$ respectively. And

$$\|P\| = \max(\|P_1\|, \|P_2\|, \|P_3\|)$$

(ii) (x_k, y_k, z_k) is an arbitrary point in a sub-rectangular box B_k



(iii) $\Delta V_k = \mathrm{Vol}(B_k) = \Delta x_k \Delta y_k \Delta z_k$.

Thm 4 (Fubini's Theorem for Triple Integrals (1st form))

If $f(x,y,z)$ is continuous (in fact, "absolutely" integrable is sufficient) on $B = [a,b]\times[c,d]\times[r,s]$, then

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_c^d \int_a^b f(x,y,z)\,dx\,dy\,dz$$

Note: Interchanging the order of the coordinates, we also have

$$\iiint_B f(x,y,z)\,dV = \int_r^s \int_a^b \int_c^d f(x,y,z)\,dy\,dx\,dz$$

$$= \cdots \text{ in any order of } dx,\ dy,\ dz.$$

Def 6 (Triple integral over a general region $D \subset \mathbb{R}^3$)

Let $f(x,y,z)$ be a function on a closed and bounded region $D \subset \mathbb{R}^3$. Then

$$\iiint_D f(x,y,z)\,dV \overset{\text{def}}{=\!=} \iiint_B F(x,y,z)\,dV$$

where B is a closed and bounded rectangular box containing D, and

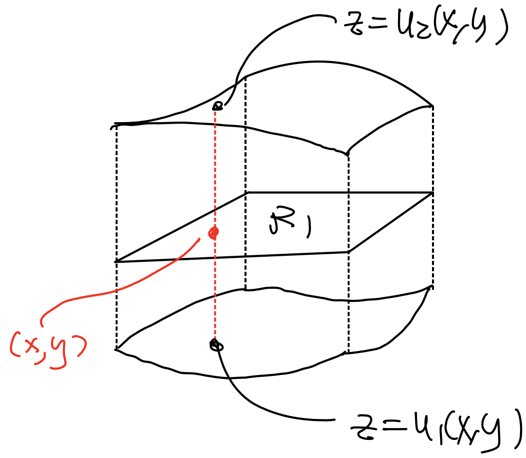
$$F(x,y,z) = \begin{cases} f(x,y,z), & \text{if } (x,y,z) \in D \\ 0, & \text{if } (x,y,z) \in B \setminus D. \end{cases}$$

Note: As in double integral, this definition is well-defined.

Special types of closed and bounded region $D \subset \mathbb{R}^3$

(1) $D = \{(x,y,z) : (x,y) \in R_1 ,\ u_1(x,y) \le z \le u_2(x,y)\}$

$(u_1(x,y) \le u_2(x,y) ,\ u_1 \not\equiv u_2)$



(2) $D = \left\{ \begin{array}{l} (x,y,z) : (x,z) \in R_2 \\ v_1(x,z) \le y \le v_2(x,z) \end{array} \right\}$

$(v_1 \le v_2 ,\ v_1 \not\equiv v_2)$

(3) $D = \{(x,y,z) : (y,z) \in R_3 ,\ w_1(y,z) \le x \le w_2(y,z)\}$

$(w_1 \le w_2 ,\ w_1 \not\equiv w_2)$

where R_i, $i=1,2,3$ are closed and bounded plane regions and u_1, u_2; v_1, v_2; w_1, w_2 are continuous wrt the corresponding variables.

Thm5 (Fubini's Thm for Triple integrals (Strong form))

Let $f(x,y,z)$ be a continuous (absolutely integrable) function on D. If D is of type (1) as above, then

$$\iiint_D f(x,y,z)\, dV = \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\, dz \right] dx\, dy$$

Similarly for types (2) and (3).

Note: Particularly, we have (using Fubini's for double integrals)

if $D = \left\{ (x,y,z) : \begin{array}{l} a \le x \le b, \ g_1(x) \le y \le g_2(x) \\ u_1(x,y) \le z \le u_2(x,y) \end{array} \right\}$

(i.e. R_1 is of type (1) as in double integrals), then

$$\boxed{\iiint_D f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx}$$

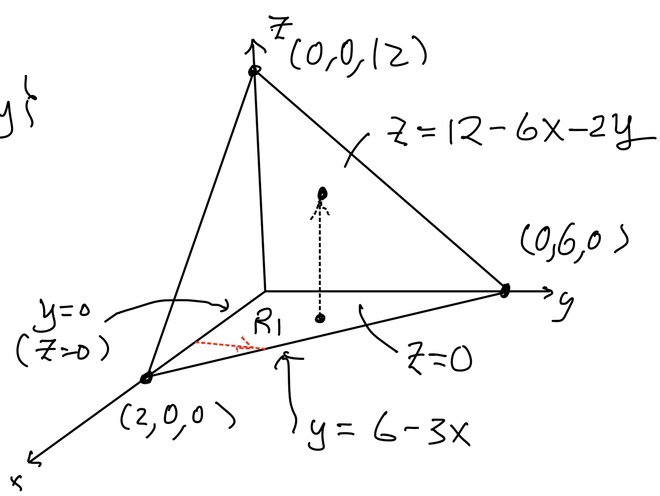
Similarly for other types.

> Prop6: The propositions 1 – 4 for double integrals also hold for triple integrals over closed and bounded region in $\mathbb{R}^3$.

eg17: Volume of the bounded region D in the 1st octant enclosed by the plane $6x+2y+z=12$

Soln: $D = \{(x,y) \in R_1 : 0 \le z \le 12-6x-2y\}$

$= \left\{ \begin{array}{l} 0 \le x \le 2, \ 0 \le y \le 6-3x \\ 0 \le z \le 12-6x-2y \end{array} \right\}$



$\Rightarrow \mathrm{Vol}(D) = \iiint_D 1\,dV$

$$= \int_0^2 \int_0^{6-3x} \int_0^{12-6x-2y} 1 \, dz \, dy \, dx$$

$$= \cdots = 24 \qquad \text{(Check!)}$$

<u>Remark</u>: For D of type 1,

$$\mathrm{Vol}(D) = \iiint_D 1 \, dV \overset{\text{Fubini}}{=\!=} \iint_{R_1} \left[\int_{u_1(x,y)}^{u_2(x,y)} 1 \, dz \right] dA$$

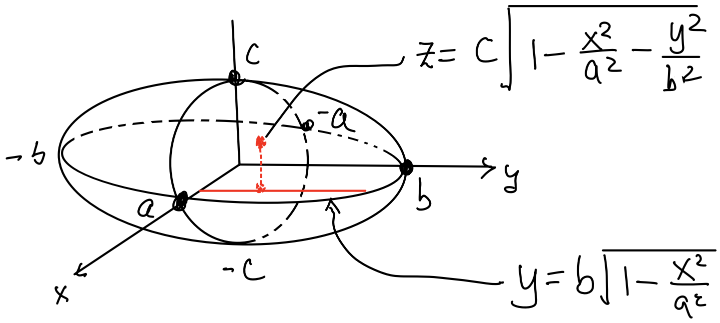
$$= \iint_{R_1} [u_2(x,y) - u_1(x,y)] \, dA$$

Formula for volume between two graphs $z = u_2(x,y)$ and $z = u_1(x,y)$.

eg18 : <u>Volume of Ellipsoid</u>

$$D = \left\{ (x,y,z) : \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \le 1 \right\} \quad (a,b,c > 0)$$

<u>Soln</u>



By symmetry, we can consider the 1st octant only and

$$\begin{aligned} Vol(D) &= 8 \cdot \text{volume of } D \text{ in the 1st octant} \\ &= 8 \int_0^a \int_0^{b\sqrt{1-\frac{x^2}{a^2}}} \int_0^{c\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}} 1 \, dz \, dy \, dx \\ &= \cdots = \frac{4\pi abc}{3} \text{ (optional exercise)} \end{aligned}$$

[In fact, we will have a better way to calculate this volume by "change of variables formula" (later)]