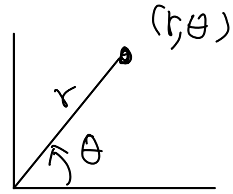
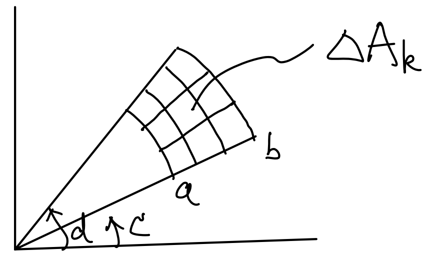
Double integral in polar coordinates



$(r,\theta) \longleftrightarrow (x,y)$

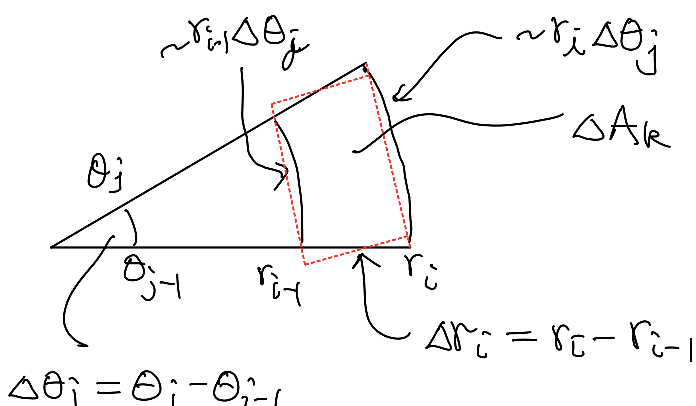
$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \end{cases}$$

$$\begin{cases} a \le r \le b \\ c \le \theta \le d \end{cases}$$



<u>Idea</u>: $\sum_k f(\text{point}_k)\,\Delta A_k$

What is ΔA_k (approximately)?



$$\therefore \quad \Delta A_k \simeq (r_i\Delta\theta_j)\cdot\Delta r_i \quad \left(\simeq (r_{i-1}\Delta\theta_j)\cdot\Delta r_i\right)$$

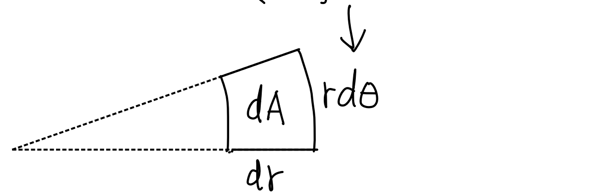
Hence $\Delta A_k \cong \Delta x\,\Delta y \simeq (r\Delta\theta)\cdot\Delta r$

So $$\iint_R f(x,y)\,dA = \iint_R f(x,y)\,\underbrace{dx\,dy}$$

$$= \iint_R f(r\cos\theta, r\sin\theta)\,\underbrace{r\,dr\,d\theta}$$

Method to remember the formula

$$dA = dx\,dy = r\,dr\,d\theta$$



Double integral of f over $R = \{(r,\theta) : a \le r \le b,\ c \le \theta \le d\}$ in polar coordinates is

$$\iint_R f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta = \int_c^d \left(\int_a^b f(r,\theta)\, r\,dr \right) d\theta$$

$$= \int_a^b \left(\int_c^d f(r,\theta)\, d\theta \right) r\,dr$$

where $f(r,\theta)$ is the simplified notation for $f(r\cos\theta, r\sin\theta)$

Remark: This is a special case of the change of variables formula. The "extra" factor "r" in the integrand is in fact

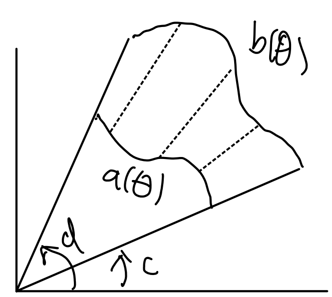
$$r = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$ the Jacobian determinant of the change of variables.

More generally

Thm3: If R is a (closed and bounded) region with

$a(\theta) \le r \le b(\theta)$ and $c \le \theta \le d$

$(0 \le a(\theta) \le b(\theta),\ a(\theta) \not\equiv b(\theta))$

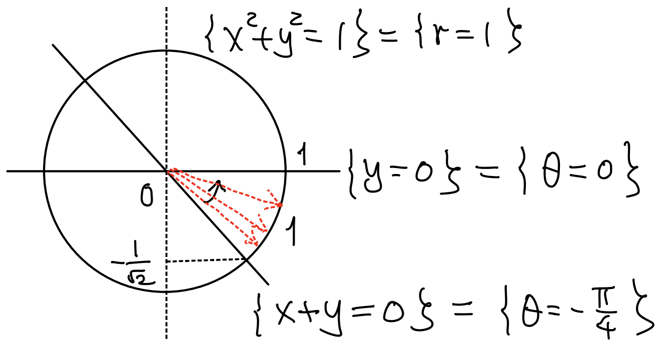


And $f : R \to \mathbb{R}$, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{a(\theta)}^{b(\theta)} f(r\cos\theta, r\sin\theta)\, r\,dr \right) d\theta$$

(remember this extra "r")

eg 12: Back to our previous example 9



$$R = \{ 0 \leq r \leq 1 , -\tfrac{\pi}{4} \leq \theta \leq 0 \}$$

$$f(x,y) = x = r\cos\theta$$

$$\iint_R x\, dA = \int_{-\frac{\pi}{4}}^{0} \left(\int_0^1 r\cos\theta \cdot r\, dr \right) d\theta = \left(\int_{-\frac{\pi}{4}}^{0} \cos\theta\, d\theta \right) \left(\int_0^1 r^2 dr \right)$$

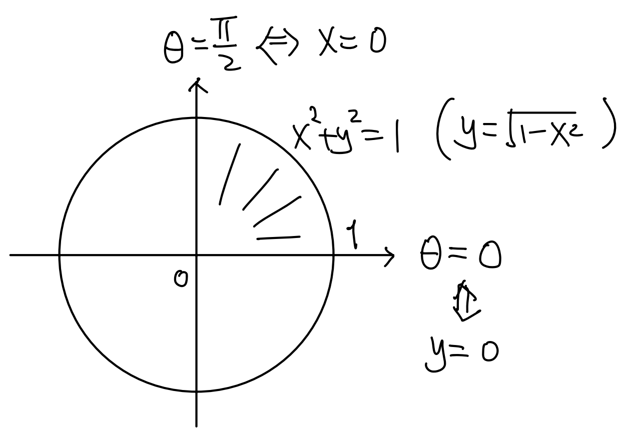
$$= \cdots = \frac{1}{3\sqrt{2}} \quad (\text{check!})$$

Much easier than before!

eg 13 Convert integrals between Cartesian and Polar coordinates

(a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta \cos\theta \, dr\, d\theta$

(b) $\int_1^2 \int_0^{\sqrt{2x-x^2}} y\, dy\, dx$



Soln: (a) $\int_0^{\frac{\pi}{2}} \int_0^1 r^3 \sin\theta\cos\theta\, dr\, d\theta$

$$= \int_0^{\frac{\pi}{2}} \int_0^1 (r\cos\theta)(r\sin\theta)\, r\, dr\, d\theta$$

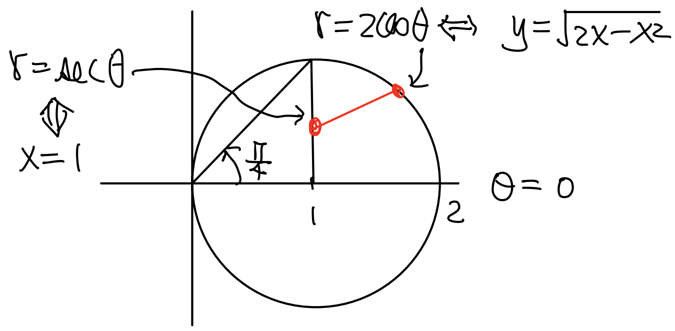
$$= \int_0^1 \int_0^{\sqrt{1-x^2}} xy\, dy\, dx$$

(b)

$$\int_1^2 \int_0^{\sqrt{2x-x^2}} y \, dy \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left[\int_{\sec\theta}^{2\cos\theta} (r\sin\theta)\, r\, dr \right] d\theta$$

$$= \int_0^{\frac{\pi}{4}} \left[\int_{\sec\theta}^{2\cos\theta} r^2 \sin\theta \, dr \right] d\theta$$



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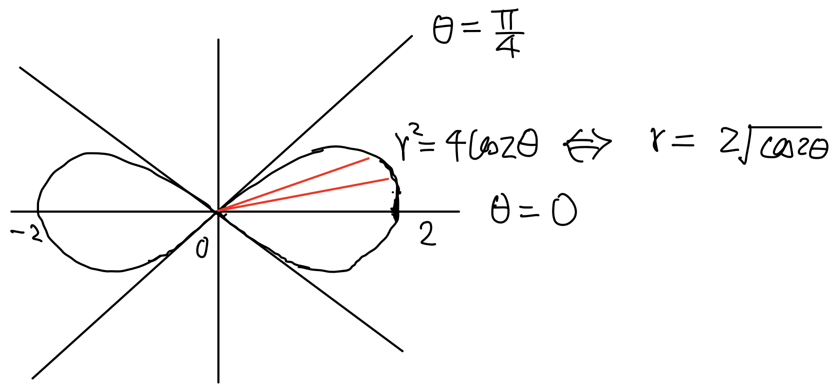
eg 14: Find area enclosed by $r^2 = 4\cos 2\theta$.

Soln:



By symmetry,

$$\text{Area} = 4\int_0^{\frac{\pi}{4}} \left(\int_0^{2\sqrt{\cos 2\theta}} 1 \cdot r\, dr \right) d\theta$$

$$= 4\int_0^{\frac{\pi}{4}} \left[\frac{r^2}{2} \right]_0^{2\sqrt{\cos 2\theta}} d\theta$$

$$= 8\int_0^{\frac{\pi}{4}} \cos 2\theta \, d\theta = 4 \quad \text{(check)}$$

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Remark: r is "not really" a function of all θ, it should be regarded as a "level set":

(i) there is no solution when $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$ & $\frac{5\pi}{4} < \theta < \frac{7\pi}{4}$

(ii) in term of (x, y) coordinates

$$F(x,y) = (x^2+y^2)^2 - 4(x^2-y^2) = 0 \quad \text{(Check!)}$$

which has a critical point at $(x,y) = (0,0)$ $\left(\vec{\nabla}F(0,0) = \vec{0}\right)$

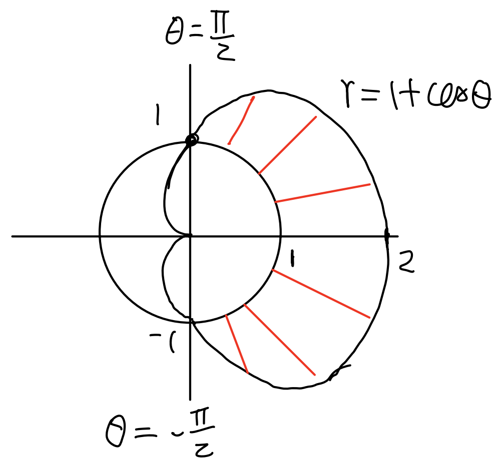
on the level set. (One cannot apply "Implicit Function Theorem" at the critical point $(0,0)$) (later for more detail)

eg 15: Integrate $f(x,y) = \dfrac{1}{\sqrt{x^2+y^2}}$ over the region R bounded between

$$\begin{cases} r = 1+\cos\theta & \text{(cardiod)} \\ r = 1 \end{cases}$$

and outside the circle $r=1$.

Soln $r = 1+\cos\theta$



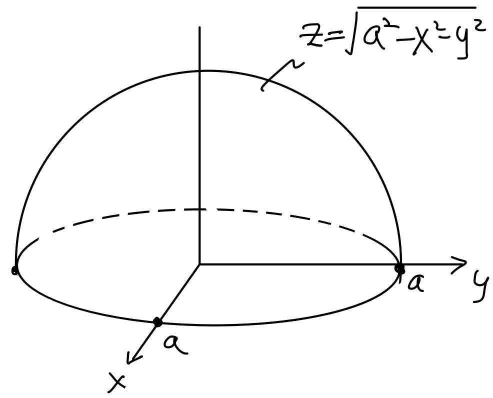
Intersection $1 = 1+\cos\theta$

$$\iint_R f(x,y)\,dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left(\int_1^{1+\cos\theta} \frac{1}{r} \cdot r\,dr \right) d\theta$$

$$= \cdots = 2 \quad \text{(Check!)}$$

eg16: Let $z = \sqrt{a^2-x^2-y^2}$ be a function defined on

$$R = \{(x,y) : x^2+y^2 \le a^2\}$$



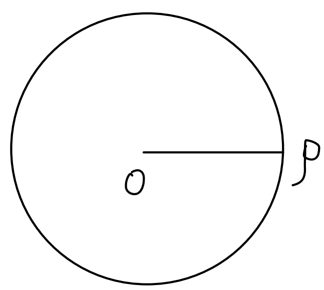
The graph of z is the (upper) hemisphere of radius a. Find the average height of the hemisphere.

Soln:
$$\text{Average height} = \frac{1}{\text{Area}(R)} \iint_R z \, dA$$
$$= \frac{1}{\pi a^2} \int_0^{2\pi} \left(\int_0^a \sqrt{a^2-r^2} \cdot r dr \right) d\theta$$
$$= \frac{2a}{3} \quad \text{(check!)}$$

eg17 (Improper integral)

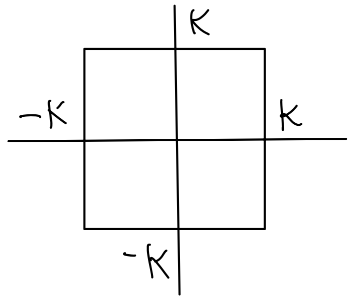
Find $\int_{-\infty}^{\infty} e^{-x^2} dx$

Soln: Consider $\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA$ (Also an improper integral)



$$= \lim_{\rho \to +\infty} \iint_{\{x^2+y^2 \le \rho^2\}} e^{-(x^2+y^2)} dA$$
$$= \lim_{\rho \to +\infty} \int_0^{2\pi} \left(\int_0^{\rho} e^{-r^2} \cdot r dr \right) d\theta$$
$$= \lim_{\rho \to +\infty} \pi(1-e^{-\rho^2}) = \pi \quad \text{(check!)}$$

On the other hand



$$\iint_{\mathbb{R}^2} e^{-x^2-y^2}\, dA$$

$$= \lim_{K\to+\infty} \int_{-K}^{K}\int_{-K}^{K} e^{-x^2} e^{-y^2}\, dx\, dy$$

$$= \lim_{K\to+\infty} \left(\int_{-K}^{K} e^{-x^2} dx\right)\left(\int_{-K}^{K} e^{-y^2} dy\right)$$

$$= \lim_{K\to+\infty} \left(\int_{-K}^{K} e^{-x^2} dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx\right)^2$$

$$\Rightarrow \quad \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \quad \#$$

Caution: we are calculating $\iint_{\mathbb{R}^2} e^{-x^2-y^2} dA$ in <u>two</u> different limiting processes. Why are they equal?

<u>Hints</u>: $e^{-x^2} > 0$ and