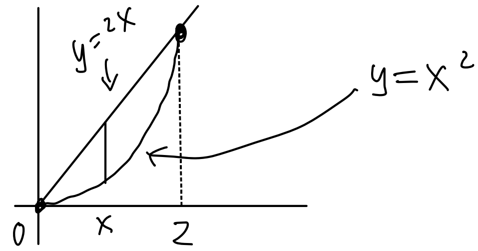
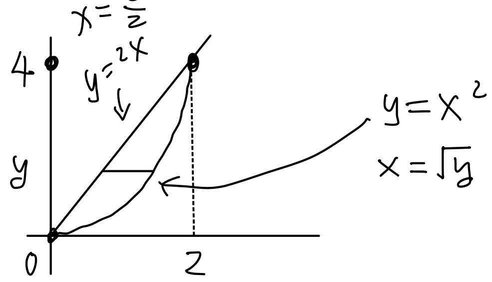
(Cont'd)



is type (1): $R = \{0 \le x \le 2,\ x^2 \le y \le 2x\}$

$$\iint_R f(x,y)\,dA = \int_0^2 \left[\int_{x^2}^{2x} (4y+2)\,dy\right] dx = \int_0^2 (-2x^4+6x^2+4x)\,dx$$
$$= \frac{56}{5} \qquad \text{(check!)}$$

Or

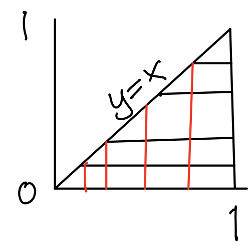


(Also type (2))

$$\iint_R f(x,y)\,dA = \int_0^4 \left[\int_{\frac{y}{2}}^{\sqrt{y}} (4y+2)\,dx\right] dy$$
$$= \int_0^4 (4y+2)\left(\sqrt{y}-\frac{y}{2}\right) dy$$
$$= \frac{56}{5} \quad \text{(check!)}$$

eg8: Evaluate $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$.

Soln: Regard $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy$.



as a double integral of $\frac{\sin x}{x}$

over the region $\{y \le x \le 1,\ 0 \le y \le 1\}$

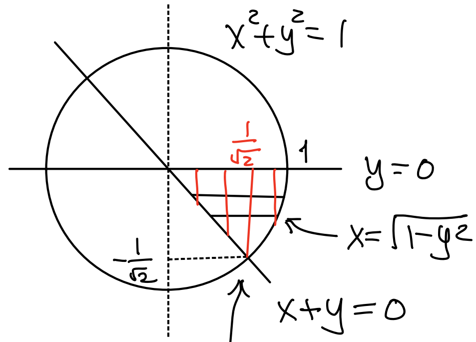
By Fubini's Thm $\displaystyle\int_0^1\int_y^1 \frac{\sin x}{x}\,dx\,dy = \int_0^1\int_0^x \frac{\sin x}{x}\,dy\,dx = \int_0^1 \sin x\,dx$

$= 1 - \cos 1$. ※

(Caution: $f(x,y) = \frac{\sin x}{x}$ doesn't define at $x=0$. Why Fubini applied?

(Hint: $\lim_{x\to 0} \frac{\sin x}{x} = 1$))

eg 9: Find $\iint_R x\,dA$, where R is the region in the right half-plane bounded by $y=0$, $x+y=0$, and the unit circle.



need to solve eqts $\begin{cases} x^2+y^2=1 \\ x+y=0 \end{cases}$ for the intersection point.

Soln: By Fubini's Thm,

$$\iint_R x\,dA = \int_{-\frac{1}{\sqrt{2}}}^{0}\left(\int_{-y}^{\sqrt{1-y^2}} x\,dx\right)dy$$

$$= \int_{-\frac{1}{\sqrt{2}}}^{0} \left(\frac{1}{2}-y^2\right)dy = \frac{1}{3\sqrt{2}} \quad \text{(Check!)}$$

Alternatively

$$\iint_R x\,dA = \int_0^{\frac{1}{\sqrt{2}}}\left(\int_{-x}^{0} x\,dy\right)dx + \int_{\frac{1}{\sqrt{2}}}^{1}\left(\int_{-\sqrt{1-x^2}}^{0} x\,dy\right)dx$$

$$= \int_0^{\frac{1}{\sqrt{2}}} x^2\,dx + \int_{\frac{1}{\sqrt{2}}}^{1} x\sqrt{1-x^2}\,dx = \frac{1}{3\sqrt{2}} \quad \text{(Check!)}$$

Applications

(1) Area (of "good" region $R \subset \mathbb{R}^2$)

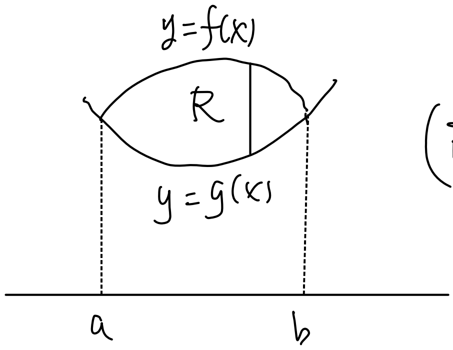
$$\boxed{\text{Def 3}: \quad \text{Area}(R) = \iint_R 1\, dA}$$

Then Fubini's Thm implies the well-known formula

$$\text{Area}(R) = \int_a^b [f(x) - g(x)]\, dx$$

if R is the region bounded by the curves

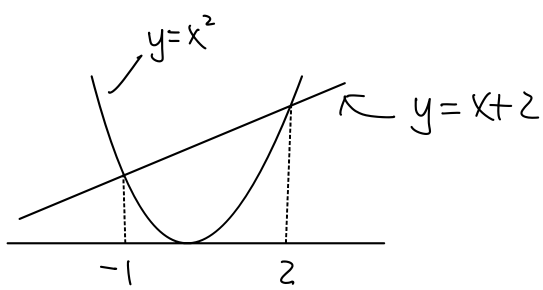
$y = f(x)$ and $y = g(x)$ for $a \le x \le b$ (with $\begin{cases} f(a) = g(a), \\ f(b) = g(b), \\ g(x) \le f(x) \end{cases}$)



(Ex!)

eg 10 Area bounded by $y = x^2$ and $y = x+2$

Soln: Solve $\begin{cases} y = x^2 \\ y = x+2 \end{cases} \Rightarrow x = -1, 2$



Then Fubini's

$$\text{Area} = \int_{-1}^{2} (x+2-x^2)\, dx = \frac{9}{2} \quad (\text{check!})$$

(2) Average (of a function over a region)

Let $f : R \subset \mathbb{R}^2 \to \mathbb{R}$ be an integrable function

$\boxed{\begin{array}{l}\underline{\text{Def 4}}: \text{ The } \underline{\text{average value}} \text{ of } f \text{ over } R \\ \qquad = \dfrac{1}{\text{Area}(R)} \displaystyle\iint_R f(x,y)\, dA\end{array}}$

eg 11 Let $f(x,y) = x\cos(xy)$, $R = [0,\pi] \times [0,1]$

Find average of f over R.

Soln: Average of f over $R = \dfrac{1}{\text{Area}(R)} \displaystyle\iint_R f(x,y)\, dA$

$$= \frac{1}{\pi} \int_0^{\pi} \int_0^1 x\cos(xy)\, dy\, dx$$

$$= \frac{1}{\pi} \int_0^{\pi} \sin x\, dx = \frac{2}{\pi} \quad (\text{check!})$$