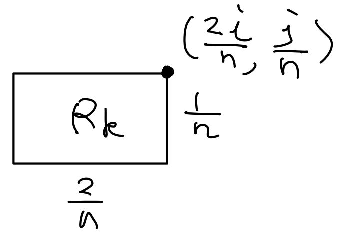
(Cont'd)

One may choose the point

$$(x_k, y_k) = \left(\frac{2i}{n}, \frac{j}{n}\right) \in R_k$$



and consider the Riemann sum

$$\sum_k f(x_k, y_k)\,\Delta A_k = \sum_{i,j=1}^{n} \frac{2i}{n}\left(\frac{j}{n}\right)^2 \cdot \frac{2}{n}\cdot\frac{1}{n}$$

$$= \frac{4}{n^5}\sum_{i,j=1}^{n} i j^2$$

$$= \frac{4}{n^5}\left(\sum_{i=1}^{n} i\right)\left(\sum_{j=1}^{n} j^2\right)$$

$$= \frac{4}{n^5}\cdot\frac{n(n+1)}{2}\cdot\frac{n(n+1)(2n+1)}{6}$$

$$\to \frac{4\cdot 2}{2\cdot 6} = \frac{2}{3} \quad \text{as } n\to\infty$$

$$\therefore \iint_{[0,2]\times[0,1]} xy^2\,dx\,dy = \frac{2}{3} \quad \#$$

Very tedious calculation.

Hence we need the following Theorem:

> **Thm 1** (Fubini's Theorem (1st form))
>
> If $f(x,y)$ is continuous on $R=[a,b]\times[c,d]$, then
>
> $$\iint_R f(x,y)\,dA = \int_c^d\left[\int_a^b f(x,y)\,dx\right]dy = \int_a^b\left[\int_c^d f(x,y)\,dy\right]dx$$

The last 2 integrals above are called <u>iterated integrals</u>

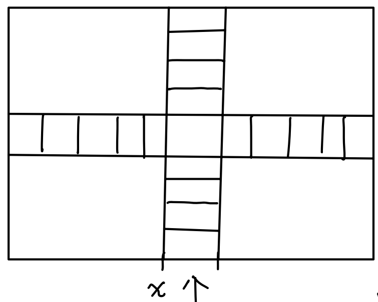
(Pf: Omitted)

<u>Ideas</u>:

sum horizontally first & taking limit

$\rightarrow \int_c^d \left[\int_a^b f(x,y)\,dx\right] dy$

("$\sum_i$" 1st in eg2)



sum vertically first & taking limit

$\rightarrow \int_a^b \left[\int_c^d f(x,y)\,dy\right] dx$ ("$\sum_j$" 1st in eg2)

<u>eg 3</u>: Using Fubini to calculate $\iint_R xy^2\,dx\,dy$, $R=[0,2]\times[0,1]$

<u>Sol</u>: By Fubini

$$\iint_R xy^2\,dA = \int_0^2\left(\int_0^1 xy^2\,dy\right)dx$$

$$= \int_0^2\left(x\int_0^1 y^2\,dy\right)dx \quad \left(= \int_0^2 \frac{x}{3}\,dx\right)$$

$$= \left(\int_0^2 x\,dx\right)\left(\int_0^1 y^2\,dy\right)$$

$$= \frac{2}{3}$$

$\#$

(Much easier than eg2)

eg4 : Sometimes the "order" of the iterated integrals is important in practice :

Find $\displaystyle\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA$.

Soln : $$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{\pi}\left[\int_0^1 x\sin(xy)\,dx\right]dy$$

$$= \int_0^{\pi}\left(-\frac{\cos y}{y} + \frac{\sin y}{y^2}\right)dy \quad \text{(integration-by-part)}$$

Not easy to integrate !

On the other hand,

$$\iint_{[0,1]\times[0,\pi]} x\sin(xy)\,dA = \int_0^{1}\left[\int_0^{\pi} x\sin(xy)\,dy\right]dx$$

$$= \int_0^1 (-\cos\pi x + 1)\,dx$$

$$= 1 \qquad \text{(easy!)} \quad \#$$

Caution : Not all functions are integrable over a (closed) rectangle.

Remark : • To show "integrable", needs to show that for all partitions and for all points (x_k, y_k) in the subrectangles, the Riemann sum

$$S(f,P) \to \text{the same number} \quad (\text{as } \|P\| \to 0)$$

- To <u>disprove</u> "integrable", needs to find, for examples

(i) <u>Some</u> P with <u>some</u> choose of (x_k, y_k) such that

$$\lim_{\|P\|\to 0} S(f,P) \text{ doesn't exist.}$$

(ii) <u>Some</u> P with <u>different</u> (x_k, y_k) & (x_k', y_k') such that

$$\underset{\text{with } (x_k, y_k)}{S(f,P)} \to a \neq b \leftarrow \underset{\text{with } (x_k', y_k')}{S'(f,P)}$$

<u>eg5</u>: Let $R=[0,1]\times[0,1]$

$$f(x,y) = \begin{cases} 0, & \text{if both } x \text{ and } y \text{ are rational} \\ 1, & \text{otherwise.} \end{cases}$$

Then f is <u>not</u> integrable over R. (using (ii))

<u>Soln</u>: $\forall$ partition P of $R = R_1 \cup \cdots \cup R_n$ (R_k subrectangles)

One can find points $(x_k, y_k) \in R_k$, $\forall k$, such that

both x_k, y_k are rational (why?)

The corresponding Riemann sum equals

$$S_n(f,P) = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k = \sum_{k=1}^{n} 0\cdot\Delta A_k = 0$$

$$\to 0, \text{ as } \|P\|\to 0$$

On the other hand, one can find points

$(x_k', y_k') \in R_k$, $\forall k$, such that at least one of the

x_k', y_k' is irrational (why?)

The corresponding Riemann sum equals

$$S_n'(f,P) = \sum_{k=1}^{n} f(x_k', y_k')\,\Delta A_k = \sum_{k=1}^{n} 1 \cdot \Delta A_k = 1$$

$$\to 1, \quad \text{as } \|P\| \to 0$$

Since $S_n(f,P) \to 0 \neq 1 \leftarrow S_n'(f,P)$,

f is not integrable. ※

<u>eg 6</u>: Let $R = [0,1] \times [0,1]$

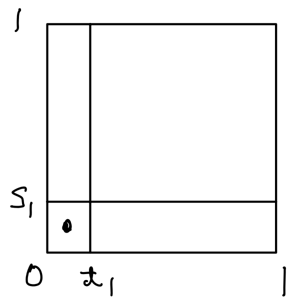
$$f(x,y) = \begin{cases} \dfrac{1}{xy}, & \text{if } x \neq 0 \text{ and } y \neq 0 \\ 0, & \text{if } x = 0 \text{ or } y = 0 \end{cases} \qquad \begin{matrix}(\geq 0) \\ \text{on } R\end{matrix}$$

Then f is <u>not</u> integrable over R. (using (i))

<u>Soln</u> In any partition P of R,

there is a sub-rectangle

$R_1 = [0, t_1] \times [0, s_1]$



Choose $(x_1, y_1) = (t_1^2, s_1^2) \in R_1$ (since $0 < t_1^2 < t_1 < 1$ & $0 < s_1^2 < s_1 < 1$)

Then Riemann sum

$$S(f,P) = \sum_{k=1}^{n} f(x_k, y_k)\,\Delta A_k$$

$$= f(x_1,y_1)\,\Delta A_1 + \sum_{k=2}^{n} f(x_k,y_k)\,\Delta A_k$$

$$\geq f(t_1^2, s_1^2)\, t_1 s_1 = \frac{1}{t_1^2 s_1^2} \cdot t_1 s_1 = \frac{1}{t_1 s_1}$$

Since $0 < t_1, s_1 \leq \|P\| \to 0$, $t_1 s_1 \to 0$

Hence $S(f,P) \geq \frac{1}{t_1 s_1} \to \infty$ as $\|P\| \to 0$

$\therefore$ Limit doesn't exist, & f is not integrable #

Remark: Egs 5 & 6 show that we need "condition(s)" to ensure the integrability of a function over closed rectangle.

Prop 1: Let $R = [a,b] \times [c,d]$ be a closed rectangle, and $f(x,y)$ be an integrable function over R, then f is bounded on R. (i.e. $\exists M > 0$ such that "$\lvert f(x,y) \rvert \leq M, \forall (x,y) \in R$.")

Pf: Omitted (eg 6 above gives an idea of proof.)

Remark: From eg 5, "boundedness" is necessary, but not sufficient for integrability.

$$\boxed{\begin{array}{c} \text{integrable} \Rightarrow \text{bounded} \\ \not\Leftarrow \\ (\uparrow \text{ in general}) \end{array}}$$

Prop 2: Let $R=[a,b]\times[c,d]$ be a closed rectangle, and $f(x,y)$ be a continuous function on R, then f is integrable on R.

Pf: Omitted (See proof in 1-variable case in MATH 2060 for an idea of proof.)

Remarks: (i) Note that a continuous function on closed rectangle is always bounded (Props 1 & 2 are consistent) (MATH 2050 for 1-variable situation)

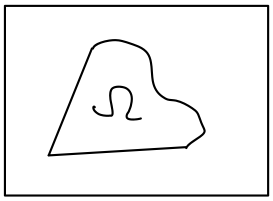
(ii) "continuity" (on closed rectangle) is sufficient, but not necessary. In fact, Prop 2 can be generalized to a bounded function on a closed rectangle with a "small" set of discontinuity. The precise concept is "measure zero set" (MATH 4050 Real Analysis). For us, we have

Prop 2' For function over closed rectangle

(a) bounded + "continuous except finitely many points"
$\Rightarrow$ integrable.

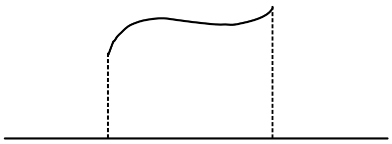
(b) bounded + "continuous except finitely many differentiable curves"
$\Rightarrow$ integrable

egs (i)



$$f(x,y) = \begin{cases} \text{continuous on } \Omega \text{ (and bounded)} \\ 0, \quad \text{otherwise (outside } \Omega) \end{cases}$$

(ii)



Ω

Furthermore, we have

<u>Prop3</u>: Let $R = [a,b] \times [c,d]$ be a <u>closed rectangle</u>, $f(x,y)$ and $g(x,y)$ be functions on R, and $k \in \mathbb{R}$ is a constant.

(1) If f & g are <u>integrable</u> over R, then $f \pm g$ and kf are <u>integrable</u> over R.

(2) In the case of (1), we have

$$\iint_R [f \pm g](x,y)\, dA = \iint_R f(x,y)\, dA \pm \iint_R g(x,y)\, dA$$

and

$$\iint_R k f(x,y)\, dA = k \iint_R f(x,y)\, dA.$$

<u>Pf</u>: Omitted (Obvious from the concept of Riemann sum.)

<u>Remark</u>: This Prop 3 implies that the set of integrable functions over (fixed) R forms a "vector space over $\mathbb{R}$", & "(double) integral" is <u>linear</u>.

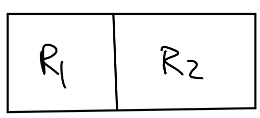
Prop 4: (a) If $f(x,y) \geq 0$ is an integrable function on a closed rectangle R, then
$$\iint_R f(x,y)\,dA \geq 0 .$$

(b) If R_1 and R_2 be two closed rectangles such that $\text{int } R_1 \cap \text{int } R_2 = \emptyset$, then
$$\iint_{R_1 \cup R_2} f(x,y)\,dA = \iint_{R_1} f(x,y)\,dA + \iint_{R_2} f(x,y)\,dA$$
for integrable function over $R_1 \cup R_2$.

Pf: Omitted (Obvious from the concept of Riemann sum)

Note: Various situations for $\text{int } R_1 \cap \text{int } R_2 = \emptyset$

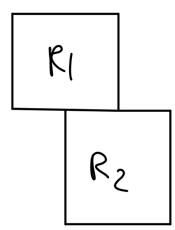
(1)



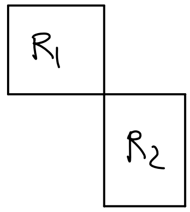
$R_1 \cap R_2 =$ common edge

$\text{int } R_1 \cap \text{int } R_2 = \emptyset$

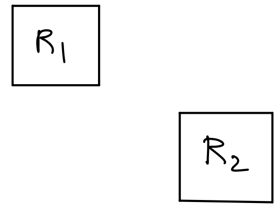
(2)



(3)



(4)

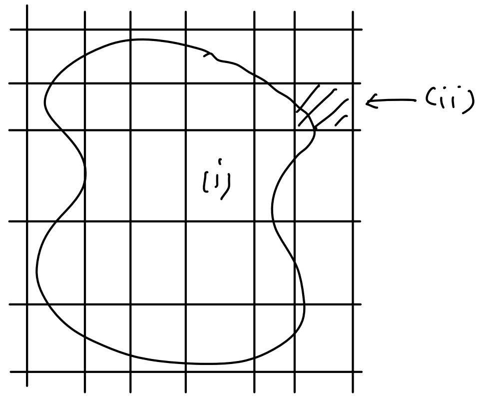


We haven't define $\iint_{R_1 \cup R_2} f(x,y)\,dA$ for cases (2) – (4) !

Hence we need to define double integrals over general regions.

Double Integrals over General Regions

For non-rectangular bounded (closed) region R, one can define similarly the concept of "Riemann sum".

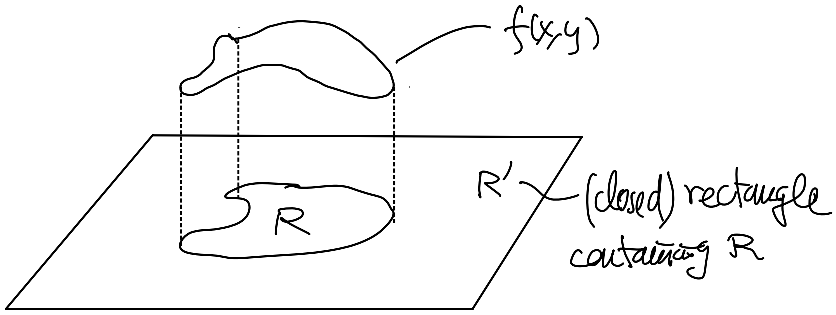


[There are two ways to form the "sum"

(i) sum over all subrectangles completely inside R

(ii) sum over all subrectangles with non-empty intersection with R.]

Or, one can define "the integrals" as follows



Def 2: Let R be a bounded region and $f(x,y)$ be a function defined on R. For any rectangle $R' \supset R$, define

$$F(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R' \setminus R \end{cases}$$

Then the integral of f over R is defined by

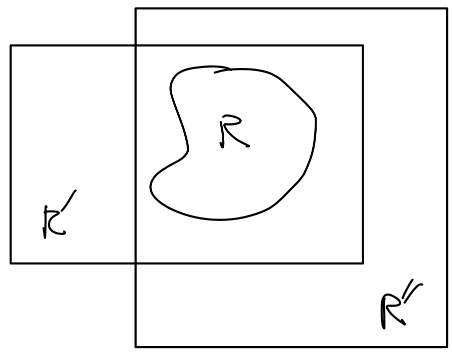
$$\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$$

Remark: The definition is well-defined (i.e. doesn't depend on the choice of R'): If R'' is another rectangle s.t. $R'' \supset R$ and

$$\tilde{F}(x,y) = \begin{cases} f(x,y), & (x,y) \in R \\ 0, & (x,y) \in R'' \setminus R \end{cases}$$

Then $\iint_{R''} \tilde{F}(x,y)\,dA = \iint_{R'} F(x,y)\,dA$

(by Prop 4 (b))



> Prop 5: The propositions 1–4 hold if we replace "closed rectangle" by "closed and bounded region"

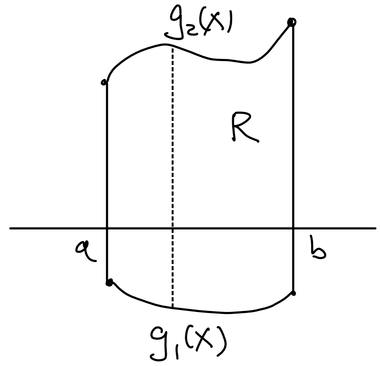
(together with the Prop 2')

Important special types of bounded regions R

Type (1): $R = \{(x,y) : a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$

where g_1 and g_2 are "continuous" functions on $[a,b]$.

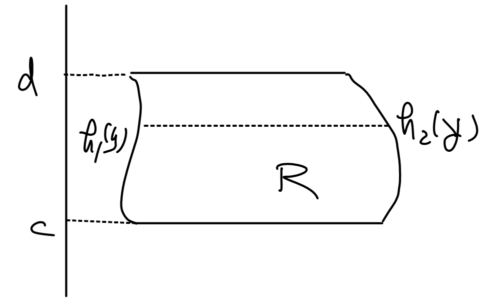
($g_1 \le g_2$, but $g_1 \not\equiv g_2$)



Type (2) : $R = \{(x,y) : h_1(y) \le x \le h_2(y),\ c \le y \le d\}$

where h_1 and h_2 are "continuous" functions on $[c,d]$

$(h_1 \le h_2$, but $h_1 \not\equiv h_2)$



For these 2 types of bounded regions, we have

<u>Thm 2 (Fubini's Thm (Stronger version))</u>

Let $f(x,y)$ be a <u>continuous</u> function on a <u>closed and bounded</u> region R.

(1) If R is of type (1) as above, then

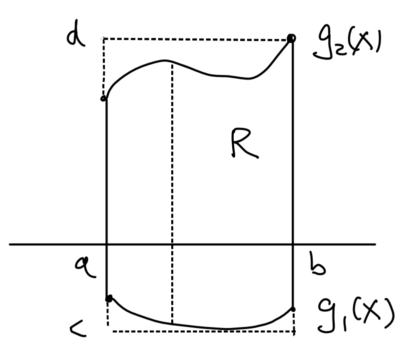
$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

(2) If R is of type (2) as above, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \right) dy = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x,y)\,dx\,dy$$

<u>Pf</u>: Type (1) : Extend $f(x,y)$ to $F(x,y)$ as in the definition on the rectangle $R' = [a,b] \times [c,d]$ such that

$$c = \min_{[a,b]} g_1(x), \quad d = \max_{[a,b]} g_2(x)$$



By definition 2, $\displaystyle\iint_R f(x,y)\,dA = \iint_{R'} F(x,y)\,dA$

$$= \int_a^b \left(\int_c^d F(x,y)\,dy \right) dx \qquad \left(\text{Fubini } (1^{st} \text{ form})\right)$$

[f continuous on $R \Rightarrow F$ continuous on R' except possibly on the boundary (curve(s)) of R. Hence by Prop 2', F (in fact $|F|$) is integrable over R'. And the Fubini theorem (1^{st} form) is in fact true for "absolutely" integrable functions on a rectangle.]

Now $F(x,y) = 0$ for $y < g_1(x)$ and $y > g_2(x)$,

and $F(x,y) = f(x,y)$ for $g_1(x) \le y \le g_2(x)$.

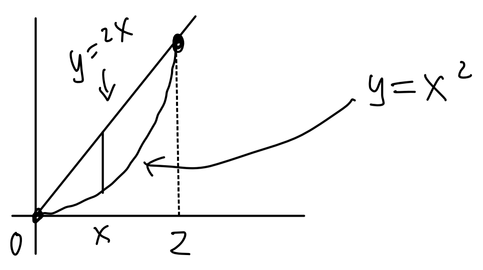
$$\therefore \iint_R f(x,y)\,dA = \int_a^b \left(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \right) dx.$$

Type (2) can be proved similarly. ⌘

<u>eg 7</u> Integrate $f(x,y) = 4y+2$

over the region bounded by $y = x^2$ and $y = 2x$.

Soln:



is type (1): $R = \{0 \le x \le 2,\ x^2 \le y \le 2x\}$

(to be cont'd)