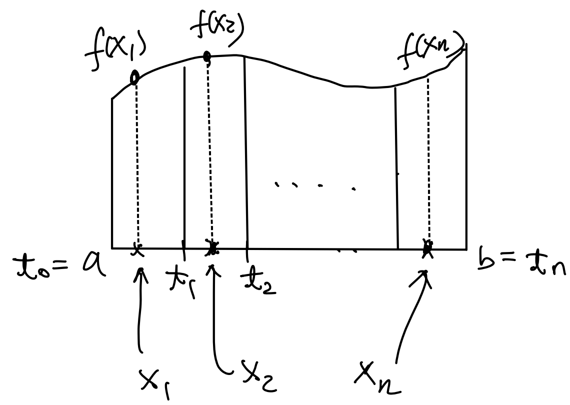
Double Integrals

Recall: In one-variable, "integral" is regarded as "limit" of "Riemann sum" (take MATH2060 for rigorous treatment)

$$\int_a^b f(x)\,dx = \lim_{\|P\|\to 0} \sum_{k=1}^{n} f(x_k)\Delta x_k$$

where
- f is a function on the interval $[a,b]$
- P is a partition $a = t_0 < t_1 < t_2 < \cdots < t_n = b$
- $x_k \in [t_{k-1}, t_k]$ and $\Delta x_k = t_k - t_{k-1}$
- $\|P\| = \max_k |\Delta x_k|$



Remark: We usually use uniform partition P

$$a = t_0 < t_1 = a + \frac{1}{n}(b-a) < t_2 = a + \frac{2}{n}(b-a) < \cdots$$

$$\cdots < t_k = a + \frac{k}{n}(b-a) < \cdots = t_n = b$$

In this case, $\|P\| = \max_k |\Delta x_k| = \dfrac{b-a}{n} \to 0$ as $n \to \infty$

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\Delta x_k = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k)\cdot \frac{b-a}{n}$$

eg1: Find $\int_0^1 x^3\,dx$ (i.e. $f(x)=x^3$ on $[0,1]$)

Soln:

(1) One may choose $x_k = \frac{k-1}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$

then
$$S_n = \sum_{k=1}^{n} f(x_k)\,\Delta x_k$$
$$= \sum_{k=1}^{n} \left(\frac{k-1}{n}\right)^3 \cdot \frac{1}{n} - \cdots \qquad \text{(Check)}$$
$$= \frac{1}{4}\left(1-\frac{1}{n}\right)^2 \to \frac{1}{4} \quad \text{as } n \to \infty$$

$$\therefore \int_0^1 x^3\,dx = \frac{1}{4}$$

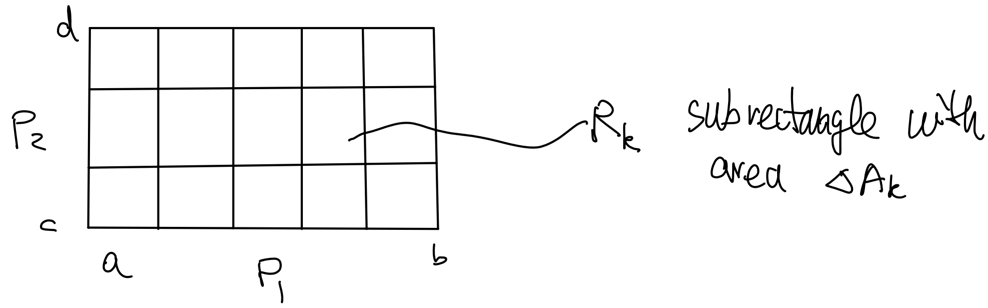
(2) Or, we can choose $x_k = \frac{k}{n} \in \left[\frac{k-1}{n}, \frac{k}{n}\right]$

then
$$S_n = \sum_{k=1}^{n} \left(\frac{k}{n}\right)^3 \frac{1}{n} = \cdots = \frac{1}{4}\left(1+\frac{1}{n}\right)^2$$
$$\to \frac{1}{4} \quad \text{as } n \to \infty \qquad \text{(same limit)}$$
$$\therefore \int_0^1 x^3\,dx = \frac{1}{4}.$$

<u>Remark</u>: We can use any $x_k \in [t_{k-1}, t_k]$ and still get the same $\int_0^1 x^3\,dx = \frac{1}{4}$.

This concept can be generalized to <u>any dimension</u>.

For 2-dim., let we first consider a function $f(x,y)$ defined on a (closed) <u>rectangle</u> $R = [a,b] \times [c,d] = \{(x,y) : a \le x \le b,\ c \le y \le d\}$



Then we can subdivide R into sub-rectangles by using partitions P_1 of $[a,b]$ & P_2 of $[c,d]$.

Denote $P = P_1 \times P_2$ (partition, subdivision, of R)

and $\|P\| = \max(\|P_1\|, \|P_2\|)$

Let the sub-rectangles be R_k, $k = 1, \cdots, N$ (= number of subrectangles)

with areas ΔA_k

Choose point $(x_k, y_k) \in R_k$ (for each $k = 1, \cdots, N$),

then consider the sum

$$S(f,P) = \sum_{k=1}^{N} f(x_k, y_k)\,\Delta A_k$$

Def 1: The function f is said to be integrable over R

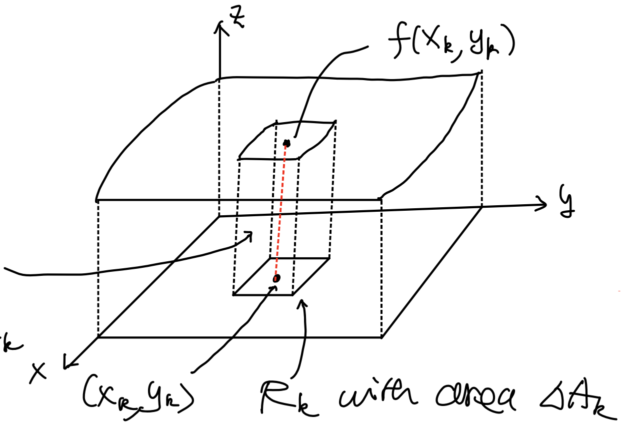
if $\displaystyle \lim_{\|P\|\to 0} S(f,P) = \lim_{\|P\|\to 0} \sum_{k=1}^{N} f(x_k,y_k)\,\Delta A_k$

exists and independent of the choice of $(x_k, y_k) \in R_k$.

In this case, the limit is called the (double) integral of f over R and is denoted by

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

Remark: Same as 1-variable, the double integral of f, $f \geq 0$, over R can be interpreted as volume under the graph of f (over R)

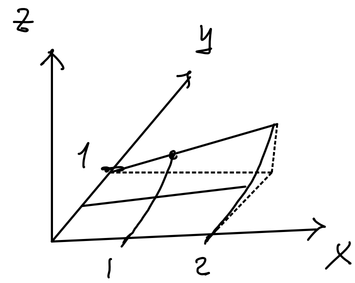


And when $f \equiv 1$,

$\iint_R 1 \, dA$ is the area of R.

eg 2: $R = [0,2] \times [0,1]$, $f(x,y) = xy^2$

(using definition) Find $\iint_R xy^2 \, dx dy$



Soln: Using uniform partitions:

$P_1 = \{0, \frac{2}{n}, \frac{4}{n}, \cdots, 2\}$ of $[0,2]$

$P_2 = \{0, \frac{1}{n}, \frac{2}{n}, \cdots, 1\}$ of $[0,1]$

$\Rightarrow$ a particular sub-rectangle is

$$R_k = \left[\frac{2(i-1)}{n}, \frac{2i}{n}\right] \times \left[\frac{j-1}{n}, \frac{j}{n}\right] \quad \text{for } \begin{array}{l} i = 1, \cdots, n \\ j = 1, \cdots, n \end{array}$$

(to be cont'd)