Tutorial 9

written by Zhiwen Zhang

Week 11

1. The first-octant portion of the cone $z = \sqrt{x^2+y^2}/2$ between the two planes $z = 0$ and $z = 3$.

Let $x = r\cos\theta$ $y = r\sin\theta$ Parametric vector form

then $z = \frac{r}{2}$

Since $x \geq 0$ $y \geq 0$ $0 \leq z \leq 3$.

We have $0 \leq r \leq 6$, $0 \leq \theta \leq \frac{\pi}{2}$

$$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + \frac{r}{2}\vec{k}$$

$$\vec{r}_r = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + \frac{1}{2}\vec{k}$$

$$\vec{r}_\theta = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j} + 0.$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & \frac{1}{2} \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -\frac{r}{2}\cos\theta\,\vec{i} - \frac{r}{2}\sin\theta\,\vec{j} + (r\cos^2\theta + r\sin^2\theta)\,\vec{k}$$

$$= -\frac{r}{2}\cos\theta\,\vec{i} - \frac{r}{2}\sin\theta\,\vec{j} + r\,\vec{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = \sqrt{\frac{r^2}{4} + r^2} = \frac{\sqrt{5}}{2}r$$

$$\iint_S f(x,y,z)\,d\sigma = \int_0^6 \int_0^{\frac{\pi}{2}} f\left(r\cos\theta, r\sin\theta, \frac{r}{2}\right)\frac{\sqrt{5}}{2}r\,d\theta\,dr$$

2. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

   The portion of the cylinder $x^2 + y^2 = 1$ between the planes $z = 1$ and $z = 4$.

$$\vec{r}(\theta, z) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + z\,\vec{k}$$

where $\quad 0 \le \theta < 2\pi \quad 1 \le z \le 4$

$$\vec{r}_\theta = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \qquad \vec{r}_z = \vec{k}$$

$$\vec{r}_\theta \times \vec{r}_z = \begin{vmatrix} i & j & k \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \cos\theta\, i + \sin\theta\, j$$

so $\quad |\vec{r}_\theta \times \vec{r}_z| = 1$

$$A = \int_0^{2\pi}\int_1^4 |\vec{r}_\theta \times \vec{r}_z|\, dz\, d\theta = \int_0^{2\pi}\int_1^4 1\, dz\, d\theta = 6\pi$$

3. Find the area of the surface cut from the paraboloid $x^2 + y^2 - z = 0$ by the plane $z = 2$.

$z = x^2 + y^2$

Let $x = r\cos\theta$ $\quad y = r\sin\theta \quad z = r^2 < 2$

So $0 \le \theta < 2\pi \quad 0 \le r \le \sqrt{2}$

$$\vec{r}(r,\theta) = r\cos\theta\,\vec{i} + r\sin\theta\,\vec{j} + r^2\,\vec{k}$$

$$\vec{r}_r(r,\theta) = \cos\theta\,\vec{i} + \sin\theta\,\vec{j} + 2r\,\vec{k}$$

$$\vec{r}_\theta(r,\theta) = -r\sin\theta\,\vec{i} + r\cos\theta\,\vec{j}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = -2r^2\cos\theta\,\vec{i} - (2r^2\sin\theta)\,\vec{j} + r\,\vec{k}$$

$$|\vec{r}_r \times \vec{r}_\theta| = (4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2)^{\frac{1}{2}}$$

$$= r\sqrt{4r^2+1}$$

$$A = \int_0^{2\pi}\int_0^{\sqrt{2}} r\sqrt{4r^2+1}\,dr\,d\theta = \frac{1}{8}\int_0^{2\pi}\int_0^{\sqrt{2}}\sqrt{4r^2+1}\,d(4r^2+1)\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi}\int_1^9 \sqrt{t}\,dt\,d\theta$$

$$= \frac{1}{8}\int_0^{2\pi} \frac{2}{3}t^{\frac{3}{2}}\Big|_1^9\,d\theta$$

$$= \frac{\pi}{4}\left(\frac{2}{3}\times 27 - \frac{2}{3}\right]$$

$$= \frac{13}{3}\pi$$

4. Let S be the surface obtained by rotating the smooth curve $y = f(x)$, $a \leqslant x \leqslant b$, about the x-axis, where $f(x) \geqslant 0$.

   (a) Show that the vector function

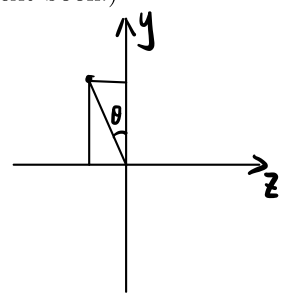
   $$\vec{r}(x, \theta) = x\vec{i} + f(x)\cos\theta\vec{j} + f(x)\sin\theta\vec{k}$$

   is a parametrization of S, where θ is the angle of rotation around the x-axis.

   (b) Show that the surface area of this surface of revolution is given by

   $$A = \int_a^b 2\pi f(x)\sqrt{1 + [f'(x)]^2}dx$$

(The figure is on Page 1003 of the text book.)



(a) $y^2 + z^2 = f(x)^2$

So $y = f(x)\cos\theta$

$z = f(x)\sin\theta$

(b) $\vec{r}(x,\theta) = x\vec{i} + f(x)\cos\theta\,\vec{j} + f(x)\sin\theta\,\vec{k}$

$\vec{r}_x = \vec{i} + f'(x)\cos\theta\,\vec{j} + f'(x)\sin\theta\,\vec{k}$

$\vec{r}_\theta = -f(x)\sin\theta\,\vec{j} + f(x)\cos\theta\,\vec{k}$

$$\vec{r}_x \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ 1 & f'(x)\cos\theta & f'(x)\sin\theta \\ 0 & -f(x)\sin\theta & f(x)\cos\theta \end{vmatrix} = f'(x)f(x)\,\vec{i} - f(x)\cos\theta\,\vec{j} - f(x)\sin\theta\,\vec{k}$$

$|\vec{r}_x \times \vec{r}_\theta| = \left([f'(x)f(x)]^2 + f^2(x)\cos^2\theta + f^2(x)\sin^2\theta\right)^{\frac{1}{2}}$

$= f(x)\left(1 + f'(x)^2\right)^{\frac{1}{2}}$

A: $\int_0^{2\pi}\int_a^b |\vec{r}_x \times \vec{r}_\theta|\,dx\,d\theta = \int_a^b 2\pi f(x)\sqrt{1 + f'(x)^2}\,dx$

5. Integrate $G(x, y, z) = x\sqrt{y^2 + 4}$ over the surface cut from the parabolic cylinder $y^2 + 4z = 16$ by the planes $x = 0, x = 1$ and $z = 0$.

$\vec{r}(x,y) = x\vec{i} + y\vec{j} + (4 - \frac{1}{4}y^2)\vec{k}$ $\quad 0 \le x \le 1 \quad -4 \le y \le 4$

$\vec{r}_x = \vec{i} \qquad \vec{r}_y = \vec{j} - \frac{1}{2}y\vec{k}$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & -\frac{1}{2}y \end{vmatrix} = \frac{1}{2}y\vec{j} + \vec{k}$$

$$|\vec{r}_x \times \vec{r}_y| = \sqrt{1 + \frac{1}{4}y^2} = \frac{1}{2}\sqrt{y^2 + 4}$$

$$\iint_S G(x,y,z)\, d\sigma = \int_0^1 \int_{-4}^4 x\sqrt{y^2+4} \cdot \frac{1}{2}\sqrt{y^2+4}\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 \int_{-4}^4 x(y^2+4)\, dy\, dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{1}{3}y^3 + 4y\right]\Big|_{-4}^4 dx$$

$$= \frac{1}{2}\int_0^1 x\left[\frac{64}{3} + 16 + \frac{64}{3} + 16\right] dx$$

$$= \frac{112}{3}$$

6. Use a parametrization to find the flux $\iint_S \vec{F} \cdot \vec{n} d\sigma$ across the surface in the specified direction.

$\vec{F} = x^2\vec{j} - xz\vec{k}$ outward through the surface cut from the parabolic cylinder $y = x^2$, $-1 \leqslant x \leqslant 1$, by the planes $z = 0$ and $z = 2$.

$\vec{r}(x,z) = x\vec{i} + x^2\vec{j} + z\vec{k} \quad -1 \leq x \leq 1 \quad 0 \leq z \leq 2$

$\vec{r}_x = i + 2x\,j \qquad \vec{r}_z = \vec{k}$

$$\vec{r}_x \times \vec{r}_z = \begin{vmatrix} i & j & k \\ 1 & 2x & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2x\vec{i} - \vec{j}$$

$\vec{F} = x^2\vec{j} - xz\vec{k}$

$$\iint_S \vec{F} \cdot \vec{n}\, d\sigma = \int_{-1}^{1}\int_0^2 \vec{F} \cdot (\vec{r}_x \times \vec{r}_z)\, dz\, dx$$

$$= \int_{-1}^{1}\int_0^2 -x^2\, dz\, dx$$

$$= 2\int_{-1}^{1} -x^2\, dx$$

$$= 2\left(-\frac{1}{3}x^3\right)\Big|_{-1}^{1}$$

$$= -\frac{4}{3}$$