Tutorial 6

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7th week

1. Find the line integral of $f(x,y,z) = \sqrt{3}/\left(x^2+y^2+z^2\right)$ over the curve $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, 1 \leq t \leq \infty$.

velocity vector $|v(t)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$

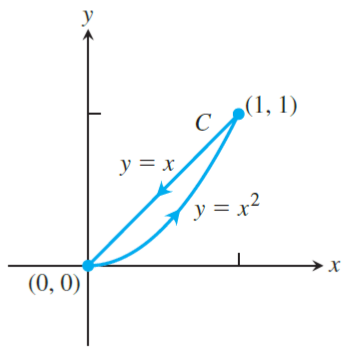
$$\int_C f(x,y,z)\,ds = \int_1^\infty f(t,t,t)\cdot |v(t)|\,dt$$

$$= \int_1^\infty \frac{\sqrt{3}}{t^2+t^2+t^2}\cdot\sqrt{3}\,dt$$

$$= \int_1^\infty \frac{1}{t^2}\,dt$$

$$= -\frac{1}{t}\Big|_1^\infty = 1$$

2. Evaluate $\int_C (x+\sqrt{y})ds$ where C is given in the accompanying figure.



C_1: curve segment following $y=x^2$ from $(0,0)$ to $(1,1)$.

C_2: line segment following $y=x$ from $(1,1)$ to $(0,0)$

We separate C into two parts, C_1 and C_2

C_1: $t\,i + t^2 j \qquad 0 \le t \le 1 \qquad x_1(t) = t \qquad y_1(t) = t^2$

C_2: $(1-t)i + (1-t)j \qquad 0 \le t \le 1 \qquad x_2(t) = 1-t \qquad y_2(t) = 1-t$

$$\int_C f(x,y)ds = \int_{C_1} f(x,y)\,|v_1(t)|\,ds + \int_{C_2} f(x,y)\,|v_2(t)|\,ds$$

$$= \int_0^1 (t+t)\sqrt{1^2+(2t)^2}\,dt + \int_0^1 \left[(1-t)+\sqrt{1-t}\right]\sqrt{1^2+1^2}\,dt$$

Let $s = 1-t$, $u = t^2$

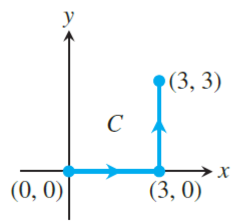
$$= 2\int_0^1 t\sqrt{1+4t^2}\,dt + \sqrt{2}\int_1^0 s+\sqrt{s}\,d(1-s)$$

$$= \int_0^1 \sqrt{1+4u}\,du + \sqrt{2}\int_0^1 s\,ds + \sqrt{2}\int_0^1 s^{\frac{1}{2}}\,ds$$

$$= \frac{1}{6}(1+4u)^{\frac{3}{2}}\Big|_0^1 + \frac{\sqrt{2}}{2}s^2\Big|_0^1 + \frac{2\sqrt{2}}{3}s^{\frac{3}{2}}\Big|_{s=0}^{s=1}$$

$$= \frac{5^{\frac{3}{2}}}{6} + \frac{\sqrt{2}}{2} + \frac{2\sqrt{2}}{3} = \frac{5\sqrt{5}+7\sqrt{2}}{6}$$

3. $\int_C (x^2 + y^2)\, dy$, where C is given in the accompanying figure.



$C_1 : \ t\,i \qquad 0 \le t \le 3 \qquad x_1 = t \quad y_1 = 0$

$C_2 : \ 3i + tj \qquad 0 \le t \le 3 \qquad x_2 = 3 \quad y_2 = t$

$$\int_C x^2 + y^2\, dy = \int_{C_1} (x^2+y^2)\, dy + \int_{C_2} (x^2+y^2)\, dy$$

$$= \int_0^3 t^2 \cdot dy_1 \quad + \int_0^3 (3^2 + t^2)\, dy_2$$

$$= \int_0^3 (9 + t^2)\, dt$$

$$= \left(9t + \tfrac{1}{3}t^3\right)\Big|_0^3$$

$$= 36$$

4. Find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

$$\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} + (x+y)\mathbf{k}$$
$$\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/6)\mathbf{k}, \quad 0 \le t \le 2\pi$$

$x(t) = \cos t \quad y(t) = \sin t \quad z(t) = \frac{t}{6}$

$$\int_C F(r(t))\,dr = \int_0^{2\pi} [2\sin t\,\vec{i} + 3\cos t\,\vec{j} + (\sin t + \cos t)\boxed{\vec{k}}]$$
$$(-\sin t\,\vec{i} + \cos t\,\vec{j} + \tfrac{1}{6}\vec{k})\,dt.$$
$$= \int_0^{2\pi} (-2\sin^2 t + 3\cos^2 t + \tfrac{1}{6}(\sin t + \cos t))\,dt$$
$$= \int_0^{2\pi} (\cos 2t - 1 + 3\,\frac{\cos 2t + 1}{2} + \frac{1}{6}\sin t + \frac{1}{6}\cos t)\,dt$$
$$= \left[\frac{1}{2}\sin 2t - t + \frac{3}{4}\sin 2t + \frac{3}{2}t - \frac{1}{6}\cos t + \frac{1}{6}\sin t\right]\Big|_0^{2\pi}$$
$$= \pi$$

5. Find the flux of the fields

$$\mathbf{F}_1 = 2x\mathbf{i} - 3y\mathbf{j} \quad \text{and} \quad \mathbf{F}_2 = 2x\mathbf{i} + (x - y)\mathbf{j}$$

across the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi$$ $x(t) = a\cos t \quad y(t) = a\sin t.$

For F_1: $M_1 = 2a\cos t \qquad N_1 = -3a\sin t$

$$\text{Flux} = \oint_C M_1 dy - N_1 dx = \int_0^{2\pi} 2a\cos t \, d(a\sin t) - \int_0^{2\pi} -3a\sin t \, d(a\cos t)$$

$$= 2a^2\int_0^{2\pi} \cos^2 t \, dt - 3a^2\int_0^{2\pi} \sin^2 t \, dt$$

$$= 2a^2\int_0^{2\pi} \frac{\cos 2t + 1}{2} dt - 3a^2\int_0^{2\pi} \frac{1-\cos 2t}{2} dt$$

$$= 2a^2\left[\frac{1}{4}\sin 2t + \frac{1}{2}t\right]\Big|_0^{2\pi} - 3a^2\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]\Big|_0^{2\pi}$$

$$= 2a^2\pi - 3a^2\pi = -a^2\pi$$

For F_2: $M_2 = 2a\cos t \qquad N_2 = a\cos t - a\sin t$

$$\text{Flux} = \oint_C M_2 dy - N_2 dx = \int_0^{2\pi} 2a\cos t \, d(a\sin t) - \int_0^{2\pi} a\cos t - a\sin t \, d\, a\cos t$$

$$= 2a^2\int_0^{2\pi} \cos^2 t \, dt + a^2\int_0^{2\pi} \cos t \sin t \, dt - a^2\int_0^{2\pi} \sin^2 t \, dt$$

$$= 2a^2\int_0^{2\pi} \frac{1+\cos 2t}{2} dt + a^2\int_0^{2\pi} \frac{\sin 2t}{2} dt - a^2\int_0^{2\pi} \frac{1-\cos 2t}{2} dt$$

$$= 2a^2\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right]\Big|_0^{2\pi} + a^2\left[-\frac{1}{4}\cos 2t\right]\Big|_0^{2\pi} - a^2\left[\frac{1}{2}t - \frac{1}{4}\sin 2t\right]\Big|_0^{2\pi}$$

$$= 2a^2\pi - a^2\pi$$

$$= a^2\pi$$