Tutorial 3

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4th week

1. The region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$ is the base of a solid right cylinder. The top of the cylinder lies in the plane $z = x$. Find the cylinder's volume.

$1+\cos\theta \geq 1 \qquad \theta \in [-\pi, \pi) \Rightarrow \qquad \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$0 \leq z \leq x \qquad \Rightarrow \qquad 0 \leq z \leq r\cos\theta$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} \int_{0}^{r\cos\theta} dz \; r \, dr \, d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_{1}^{1+\cos\theta} r^2 \cos\theta \, dr \, d\theta.$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3}\cos\theta \, r^3 \Big|_{1}^{1+\cos\theta} dr \, d\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta \left((1+\cos\theta)^3 - 1 \right) d\theta$$

$$= \frac{1}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta + 3\cos^3\theta + 3\cos^2\theta \, d\theta$$

$$= \frac{1}{3}\left[\frac{1}{32}\sin 4\theta + \frac{1}{4}\sin 2\theta + \frac{3}{8}\theta + \frac{1}{4}\sin 3\theta + \frac{9}{4}\sin\theta + \frac{3}{2}\theta + \frac{3}{4}\sin 2\theta \right]\Bigg|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$$

$$= \frac{1}{3}\left[0 + 0 + \frac{3}{8}\pi + \frac{1}{4}(-2) + \frac{9}{4}(2) + \frac{3\pi}{2} \right].$$

$$= \frac{5}{8}\pi + \frac{4}{3}$$

$\cos 4x = \cos 2\cdot 2x$

$= 2\cos^2 2x - 1$

$= 2(2\cos^2 x - 1)^2 - 1$

$= 2(4\cos^4 x - 4\cos^2 x + 1) - 1$

$= 8\cos^4 x - 8\cos^2 x + 1$

$= 8\cos^4 x - 4(1+\cos 2x) + 1$

$= 8\cos^4 x - 4\cos 2x - 3$

$\Rightarrow \cos^4 x = \frac{1}{8}[\cos 4x + 4\cos 2x + 3]$

$\cos 3\theta = \cos(\theta + 2\theta)$

$= \cos\theta\cos 2\theta - \sin\theta\sin 2\theta$

$= \cos\theta(2\cos^2\theta - 1) - 2\sin^2\theta\cos\theta$

$= 2\cos^3\theta - \cos\theta - 2(1-\cos^2\theta)\cos\theta$

$= 4\cos^3\theta - 3\cos\theta$

$\Rightarrow \cos^3\theta = \frac{1}{4}\cos 3\theta + \frac{3}{4}\cos\theta$

$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$

2. Use the double integral in polar coordinates to derive the formula

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \mathrm{d}\theta$$

for the area of the fan-shaped region between the origin and polar curve $r = f(\theta)$, $\alpha \leqslant \theta \leqslant \beta$.

$$A = \int_{\alpha}^{\beta} \int_{0}^{f(\theta)} r \, dr \, d\theta$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \Big|_{0}^{f(\theta)} d\theta$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} f^2(\theta) \, d\theta$$

$$= \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta$$

3. Evaluate the integral

$$\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} dpdqdr$$

$$\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} \, dp \, dq \, dr$$

$$= \int_0^7 \int_0^2 \frac{q\sqrt{4-q^2}}{r+1} \, dq \, dr$$

Let $A(r) = \int_0^2 \frac{q\sqrt{4-q^2}}{r+1} \, dq$

Let $q = 2\sin\theta \qquad \theta \in [0, \frac{\pi}{2}]$.

$$A(r) = \frac{1}{r+1} \int_0^{\frac{\pi}{2}} 2\sin\theta \cdot 2\cos\theta \, d\, 2\sin\theta$$

$$= \frac{1}{r+1} \int_0^{\frac{\pi}{2}} 8 \sin\theta \cos^2\theta \, d\theta$$

$$= \frac{8}{r+1} \int_0^{\frac{\pi}{2}} \sin\theta(1-\sin^2\theta) \, d\theta$$

Note $\sin 3\theta = \sin(\theta + 2\theta) = \sin\theta\cos 2\theta + \sin 2\theta \cos\theta$

$$= \sin\theta(1 - 2\sin^2\theta) + 2\sin\theta\cos^2\theta$$

$$= \sin\theta - 2\sin^3\theta + 2\sin\theta(1-\sin^2\theta)$$

$$= 3\sin\theta - 4\sin^3\theta$$

Hence $\sin\theta(1-\sin^2\theta) = \frac{1}{4}(\sin 3\theta + \sin\theta)$

$$A(r) = \frac{8}{r+1} \int_0^{\frac{\pi}{2}} \frac{1}{4}(\sin 3\theta + \sin\theta) \, d\theta$$

$$= \frac{2}{r+1} \left[-\frac{1}{3}\cos 3\theta - \cos\theta \right] \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{2}{r+1} \cdot \frac{4}{3} = \frac{8}{3} \cdot \frac{1}{r+1}$$

Hence $\int_0^7 \int_0^2 \int_0^{\sqrt{4-q^2}} \frac{q}{r+1} \, dpdqdr = \int_0^7 A(r) dr = \frac{8}{3} \int_0^7 \frac{1}{r+1} dr$

$$= \frac{8}{3} \ln(r+1) \Big|_0^7 = 8 \ln 2$$

4. Evaluate the integral

$$\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} dydzdx$$

Integral region

$$\left.\begin{array}{l} 0 \le x \le 2 \\ 0 \le z \le 4-x^2 \\ 0 \le y \le x \end{array}\right\} \Rightarrow \left.\begin{array}{l} 0 \le z \le 4 \\ 0 \le x^2 \le 4-z \\ 0 \le y \le x \end{array}\right\} \Rightarrow \begin{array}{l} 0 \le z \le 4 \\ 0 \le x \le \sqrt{4-z} \\ 0 \le y \le x \end{array}$$

$$\int_0^2 \int_0^{4-x^2} \int_0^x \frac{\sin 2z}{4-z} \, dy \, dz \, dx$$

$$= \int_0^4 \int_0^{\sqrt{4-z}} \int_0^x \frac{\sin 2z}{4-z} \, dy \, dx \, dz$$

$$= \int_0^4 \int_0^{\sqrt{4-z}} \frac{\sin 2z}{4-z} \, x \, dx \, dz$$

$$= \int_0^4 \frac{1}{2} \frac{\sin 2z}{4-z} x^2 \Big|_{x=0}^{\sqrt{4-z}} dz$$

$$= \int_0^4 \frac{1}{2} \sin 2z \, dz$$

$$= \frac{1}{4}(-\cos 2z)\Big|_0^4 = \frac{1}{4}(-\cos 8 + 1) = \frac{\sin^2 4}{2}$$

Set up the iterated integral

5. D is the right cylinder whose base is the circle $r = 3\cos\theta$ and whose top lies in the plane $z = 5 - x$.

$r \geq 0 \Rightarrow 3\cos\theta \geq 0 \Rightarrow \theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$

$$V = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{3\cos\theta} \int_0^{5-x} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{3\cos\theta} \int_0^{5-r\cos\theta} f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

6. Let D be the region bounded below by the cone $z = \sqrt{x^2 + y^2}$ and above by the plane $z = 1$. Set up the triple integrals in spherical coordinates that give the volume of D using the following orders of integration

(a) $d\rho d\phi d\theta$

(b) $d\phi d\rho d\theta$

$x = \rho \sin\phi \cos\theta$ $\quad \theta \in [0, 2\pi)$

$y = \rho \sin\phi \sin\theta$ $\quad \phi \in [0, \pi]$

$z = \rho \cos\phi$ $\quad \rho \geq 0$

$$\sqrt{x^2+y^2} \leq z \leq 1 \Rightarrow \sqrt{\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta} \leq \rho\cos\phi \leq 1$$

$$\Rightarrow \rho\sin\phi \leq \rho\cos\phi \leq 1$$

(a) $\rho \sin\phi \leq \rho\cos\phi \Rightarrow \sin\phi \leq \cos\phi \Rightarrow \phi \in [0, \frac{\pi}{4}]$

$\rho\cos\phi \leq 1 \Rightarrow \rho \leq \frac{1}{\cos\phi}$

$$V = \int_0^{2\pi} \int_0^{\frac{\pi}{4}} \int_0^{\frac{1}{\cos\phi}} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$$

(b) $\rho\sin\phi \leq \rho\cos\phi \Rightarrow \phi \in [0, \frac{\pi}{4}]$

$\rho\cos\phi \leq 1 \Rightarrow \rho \leq \sqrt{2}$

$\phi \leq \arccos\frac{1}{\rho}$

$$V = \int_0^{2\pi} \int_0^{\sqrt{2}} \int_0^{\arccos\frac{1}{\rho}} \rho^2 \sin\phi \, d\phi \, d\rho \, d\theta$$