Tutorial 10

written by Zhiwen Zhang

Week 12

1. Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

   $\mathbf{F} = x^2\mathbf{i} + 2x\mathbf{j} + z^2\mathbf{k}$

   C: The ellipse $4x^2 + y^2 = 4$ in the xy-plane, counter-clockwise when viewed from above

$S: \vec{r}(x,y) = x\vec{i} + y\vec{j} \quad -1 \le x \le 1 \quad -\sqrt{4-4x^2} \le y \le \sqrt{4-4x^2}$

$$\nabla \times F = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ x^2 & 2x & z^2 \end{vmatrix} = \vec{i}(0) + \vec{j}(0) + \vec{k}(2) = 2\vec{k}$$

$$\vec{r}_x \times \vec{r}_y = \begin{vmatrix} i & j & k \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{vmatrix} = \vec{k}$$

$$\oint_C \vec{F} \cdot d\vec{r} = \iint_S \nabla \times \vec{F} \cdot \vec{n} \, d\sigma$$

$$= \int_{-1}^{1} \int_{-\sqrt{4-4x^2}}^{\sqrt{4-4x^2}} 2 \, dy \, dx$$

$$= 4 \int_{-1}^{1} \sqrt{4-4x^2} \, dx$$

$$= 4\pi$$

2. Let $\mathbf{n}$ be the outer unit normal of the elliptical shell

$$S : 4x^2 + 9y^2 + 36z^2 = 36, \quad z \geq 0$$

and let

$$\mathbf{F} = y\mathbf{i} + x^2\mathbf{j} + \left(x^2 + y^4\right)^{3/2} \sin e^{\sqrt{xyz}}\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n} d\sigma.$$

(Hint: One parametrization of the ellipse at the base of the shell is $x = 3\cos t, y = 2\sin t, 0 \leq t \leq 2\pi$.)

$$\iint_S \nabla \times F \cdot n \, d\sigma = \oint_C \vec{F} \, dr$$

$$C : \vec{r}(t) = 3\cos t \ \vec{i} + 2\sin t \ \vec{j} \qquad 0 \leq t < 2\pi$$

$$d\vec{r} = (-3\sin t \ \vec{i} + 2\cos t \vec{j}) \, dt$$

$$\vec{F} = 2\sin t \ \vec{i} + 9\cos^2 t \ \vec{j} + \left((3\cos t)^2 + (2\sin t)^4 \right)^{\frac{3}{2}} \sin e^{\sqrt{0}} \ \vec{k}$$

$$\iint_S \nabla \times F \cdot \vec{n} \, d\sigma = \oint_C \vec{F} \, dr$$

$$= \int_0^{2\pi} -6\sin^2 t + 18\cos^3 t \, dt$$

$$= \int_0^{2\pi} -3(1 - \cos 2t) + \frac{9}{2} (\cos 3t + 3\cos t) \, dt$$

$$= \left[-3 \left(t - \frac{\sin 2t}{2}\right) + \frac{9}{2} \left(\frac{\sin 3t}{3} + 3\sin t \right) \right]_0^{2\pi}$$

$$= -6\pi$$

3. Suppose $\mathbf{F} = \nabla \times \mathbf{A}$, where

$$\mathbf{A} = (y + \sqrt{z})\mathbf{i} + e^{xyz}\mathbf{j} + \cos(xz)\mathbf{k}.$$

Determine the flux of $\mathbf{F}$ outward through the hemisphere $x^2 + y^2 + z^2 = 1, z \geq 0$.

The boundary C is $x^2+y^2=1$ $z=0$

$C: \vec{r}(t) = \cos t \ \vec{i} + \sin t \ \vec{j}$

$d\vec{r} = (-\sin t \ \vec{i} + \cos t \ \vec{j})dt \quad \vec{A} = \sin t \ \vec{i} + \vec{j} + \vec{k}$

$$\iint_S \nabla \times \vec{A} \cdot \vec{n} \, d\sigma$$

$$= \oint_C \vec{A} \, d\vec{r}$$

$$= \int_0^{2\pi} -\sin^2 t + \cos t \, dt$$

$$= \int_0^{2\pi} -\frac{1 - \cos 2t}{2} + \cos t \, dt$$

$$= \left[-\frac{1}{2}t + \frac{1}{4}\sin 2t + \sin t \right]\Big|_0^{2\pi}$$

$$= -\pi$$

4. Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

$$\mathbf{F} = (y - z)\mathbf{i} + (z - x)\mathbf{j} + (x + z)\mathbf{k}$$

$$S: \quad \mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + \left(9 - r^2\right)\mathbf{k}$$

$$0 \le r \le 3, \quad 0 \le \theta \le 2\pi$$

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ y-z & z-x & x+z \end{vmatrix} = \vec{i}\,(-1) + \vec{j}\,(-1-1) + \vec{k}\,(-1-1) = -\vec{i} - 2\vec{j} - 2\vec{k}$$

$$\vec{r}_r \times \vec{r}_\theta = \begin{vmatrix} i & j & k \\ \cos\theta & \sin\theta & -2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (2r^2\cos\theta)\,\vec{i} + (2r^2\sin\theta)\,\vec{j} + r\,\vec{k}$$

$$\iint_S \nabla \times F \cdot \vec{n}\, d\sigma = \int_0^3 \int_0^{2\pi} (-\vec{i} - 2\vec{j} - 2\vec{k})(2r^2\cos\theta\,\vec{i} + 2r^2\sin\theta\,\vec{j} + r\vec{k})\, d\theta\, dr$$

$$= \int_0^3 \int_0^{2\pi} (-2r^2\cos\theta - 4r^2\sin\theta - 2r)\, d\theta\, dr$$

$$= -18\pi$$

5. Show that the curl of

$$\mathbf{F} = \frac{-y}{x^2+y^2}\mathbf{i} + \frac{x}{x^2+y^2}\mathbf{j} + z\mathbf{k}$$

is zero but that

$$\oint_C \mathbf{F} \cdot d\mathbf{r}$$

is not zero if C is the circle $x^2+y^2=1$ in the xy-plane. (Theorem 7 does not apply here because the domain of $\mathbf{F}$ is not simply connected. The field $\mathbf{F}$ is not defined along the z-axis so there is no way to contract C to a point without leaving the domain of $\mathbf{F}$.)

$$\nabla \times \vec{F} = \begin{vmatrix} i & j & k \\ \partial_x & \partial_y & \partial_z \\ \frac{-y}{x^2+y^2} & \frac{x}{x^2+y^2} & z \end{vmatrix} = \vec{i}(0-0) + \vec{j}(0-0) + \vec{k}\left(\partial_x \frac{x}{x^2+y^2} + \partial_y \frac{y}{x^2+y^2}\right)$$

$$\frac{\partial}{\partial x}\frac{x}{x^2+y^2} + \frac{\partial}{\partial y}\frac{y}{x^2+y^2} = \frac{x^2+y^2-2x\cdot x}{(x^2+y^2)^2} + \frac{x^2+y^2-2y\cdot y}{(x^2+y^2)^2} = 0$$

So $\nabla \times \vec{F} = \vec{0}$

C: $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$ $\quad 0 \le t < 2\pi$

$d\vec{r} = (-\sin t\,\vec{i} + \cos t\,\vec{j})dt$ $\quad \vec{F} = -\sin t\,\vec{i} + \cos t\,\vec{j}$

$$\oint_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (-\sin t\,\vec{i} + \cos t\,\vec{j})(-\sin t\,\vec{i} + \cos t\,\vec{j})\,dt$$

$$= \int_0^{2\pi} 1\,dt$$

$$= 2\pi$$