MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 9

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.5

4. Find a parametrization of the surface.

   Cone frustum The first-octant portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 4$.

   Solution. In cylindrical coordinates, let $x = r\cos\theta$, $y = r\sin\theta$, $z = 2\sqrt{x^2+y^2} \implies z = 2r$. For $2 \le z \le 4$, we have $1 \le r \le 2$. Hence, a parametrization is

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}, \qquad 1 \le r \le 2,\ 0 \le \theta \le 2\pi.$$

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10. Find a parametrization of the surface.

    Parabolic cylinder between planes The surface cut from the parabolic cylinder $y = x^2$ by the planes $z = 0$, $z = 3$, and $y = 2$.

    Solution. A parametrization is

$$\mathbf{r}(x,z) = x\mathbf{i} + x^2\mathbf{j} + z\mathbf{k}, \qquad -\sqrt{2} \le x \le \sqrt{2},\ 0 \le z \le 3.$$

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19. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

    Cone frustum The first-octant portion of the cone $z = 2\sqrt{x^2 + y^2}$ between the planes $z = 2$ and $z = 6$.

    Solution. A parametrization is

$$\mathbf{r}(r,\theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + 2r\mathbf{k}, \qquad 1 \le r \le 3,\ 0 \le \theta \le 2\pi.$$

Then $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$, and thus

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r\cos\theta)\mathbf{i} - (2r\sin\theta)\mathbf{j} + r\mathbf{k},$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^2\cos^2\theta + 4r^2\sin^2\theta + r^2} = r\sqrt{5}.$$

Hence,

$$\text{Surface area} = \int_0^{2\pi}\int_1^3 r\sqrt{5}\,dr\,d\theta. = 8\pi\sqrt{5}.$$

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22. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

Circular cylinder band The portion of the cylinder $x^2 + z^2 = 10$ between the planes $y = -1$ and $y = 1$.

Solution. A parametrization is

$$\mathbf{r}(v, y) = (\sqrt{10}\cos v)\mathbf{i} + y\mathbf{j} + (\sqrt{10}\sin v)\mathbf{k}, \qquad 0 \le v \le 2\pi,\ -1 \le y \le 1.$$

Then $\mathbf{r}_v = (-\sqrt{10}\sin v)\mathbf{i} + (\sqrt{10}\cos v)\mathbf{k}$ and $\mathbf{r}_y = \mathbf{j}$, and thus

$$\mathbf{r}_v \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sqrt{10}\sin v & 0 & \sqrt{10}\cos v \\ 0 & 1 & 0 \end{vmatrix} = (-\sqrt{10}\cos v)\mathbf{i} + (-\sqrt{10}\sin v)\mathbf{k},$$

$$|\mathbf{r}_v \times \mathbf{r}_y| = \sqrt{10}.$$

Hence,

$$\text{Surface area} = \int_0^{2\pi} \int_{-1}^{1} \sqrt{10}\, dy\, dv = 4\pi\sqrt{10}.$$

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24. Use a parametrization to express the area of the surface as a double integral. Then evaluate the integral.

Parabolic band The portion of the paraboloid $z = x^2 + y^2$ between the planes $z = 1$ and $z = 4$.

Solution. A parametrization is

$$\mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r^2\mathbf{k}, \qquad 1 \le r \le 2,\ 0 \le \theta \le 2\pi.$$

Then $\mathbf{r}_r = (\cos\theta)\mathbf{i} + (\sin\theta)\mathbf{j} + 2r\mathbf{k}$ and $\mathbf{r}_\theta = (-r\sin\theta)\mathbf{i} + (r\cos\theta)\mathbf{j}$, and thus

$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 2r \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-2r^2\cos\theta)\mathbf{i} - (2r^2\sin\theta)\mathbf{j} + r\mathbf{k},$$

$$|\mathbf{r}_r \times \mathbf{r}_\theta| = \sqrt{4r^4\cos^2\theta + 4r^4\sin^2\theta + r^2} = r\sqrt{4r^2 + 1}.$$

Hence,

$$\text{Surface area} = \int_0^{2\pi} \int_1^2 r\sqrt{4r^2 + 1}.\, dr\, d\theta. = 2\pi \cdot \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_1^2 = \frac{\pi}{6}(17\sqrt{17} - 5\sqrt{5}).$$

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38. Find the area of the band cut from the paraboloid $x^2 + y^2 - z = 0$ by the planes $z = 2$ and $z = 6$.

Solution. The band is given by the graph

$$z = f(x, y) := x^2 + y^2, \qquad (x, y) \in \Omega := \{(x, y) : 2 \le x^2 + y^2 \le 6\}.$$

Then $\nabla f = 2x\mathbf{i} + 2y\mathbf{j}$ and $|\nabla f|^2 = 4x^2 + 4y^2$. Hence,

$$\begin{aligned}\text{Surface area} &= \iint_\Omega \sqrt{1 + |\nabla f|^2}\, dA = \int_\Omega \sqrt{1 + 4x^2 + 4y^2}\, dx\, dy \\ &= \int_0^{2\pi} \int_{\sqrt{2}}^{\sqrt{6}} \sqrt{1 + 4r^2}\, r dr\, d\theta = 2\pi \left[\frac{1}{12}(4r^2 + 1)^{3/2}\right]_{\sqrt{2}}^{\sqrt{6}} = \frac{49\pi}{3}.\end{aligned}$$

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47. Find the area of the surface $x^2 - 2\ln x + \sqrt{15}y - z = 0$ above the square $R : 1 \le x \le 2$, $0 \le y \le 1$, in the xy-plane.

Solution. The surface is given by the graph

$$z = f(x, y) := x^2 - 2\ln x + \sqrt{15}y, \qquad (x, y) \in R.$$

Then $\nabla f = (2x - \frac{2}{x})\mathbf{i} + \sqrt{15}\mathbf{j}$ and

$$1 + |\nabla f|^2 = 1 + (2x - \frac{2}{x})^2 + 15 = (2x + \frac{2}{x})^2.$$

Hence,

$$\begin{aligned}\text{Surface area} &= \iint_R \sqrt{1 + |\nabla f|^2}\, dA = \int_R \sqrt{2x + \frac{2}{x})^2}\, dx\, dy \\ &= \int_0^1 \int_1^2 (2x + \frac{2}{x})\, dx\, dy = 3 + 2\ln 2.\end{aligned}$$

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Exercises 16.6

13. Integrate $G(x, y, z) = x + y + z$ over the portion of the plane $2x + 2y + z = 2$ that lies in the first octant.

Solution. The surface is given by the graph

$$z = f(x, y) := 2 - 2x - 2y, \qquad (x, y) \in \Omega := \{(x, y) : x, y \ge 0,\ x + y \le 1\}.$$

Then $\nabla f = -2\mathbf{i} - 2\mathbf{j}$ and

$$\sqrt{1 + |\nabla f|^2} = \sqrt{1 + 2^2 + 2^2} = 3.$$

Hence,

$$\begin{aligned}\iint_S G\, d\sigma &= \int_\Omega G(x, y, f(x, y))\sqrt{1 + |\nabla f|^2}\, dx\, dy \\ &= \int_0^1 \int_0^{1-x} (x + y + (2 - 2x - 2y) \cdot 3\, dy\, dx \\ &= 3\int_0^1 \int_0^{1-x} (2 - x - y)\, dy\, dx = 2.\end{aligned}$$

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19. Use a parametrization to find the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ across the surface in the specified direction.

Parabolic cylinder $\mathbf{F} = z^2\mathbf{i} + x\mathbf{j} - 3z\mathbf{k}$ outward (normal away from the x-axis) through the surface cut from the parabolic cylinder $z = 4 - y^2$ by the planes $x = 0$, $x = 1$, and $z = 0$.

Solution. A parametrization is

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (4 - y^2)\mathbf{k}, \qquad (x,y) \in \Omega := \{(x,y) : 0 \le x \le 1,\ -2 \le y \le 2\}.$$

Then $\mathbf{r}_x = \mathbf{i}$ and $\mathbf{r}_y = \mathbf{j} - 2y\mathbf{k}$, and thus

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & 0 \\ 0 & 1 & -2y \end{vmatrix} = 2y\mathbf{j} + \mathbf{k},$$

$$\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) = ((4-y^2)^2\mathbf{i} + x\mathbf{j} - 3(4-y^2)\mathbf{k}) \cdot (2y\mathbf{j} + \mathbf{k}) = 2xy - 3(4 - y^2).$$

Hence,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iint_\Omega \mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\,dx\,dy \\ &= \int_0^1 \int_{-2}^2 [2xy - 3(4 - y^2)]\,dy\,dx = -32.\end{aligned}$$

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23. Use a parametrization to find the flux $\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma$ across the surface in the specified direction.

Plane $\mathbf{F} = 2xy\mathbf{i} + 2yz\mathbf{j} + 2xz\mathbf{k}$ upward across the portion of the plane $x + y + z = 2a$ that lies above the square $0 \le x \le a$, $0 \le y \le a$, in the xy-plane.

Solution. A parametrization is

$$\mathbf{r}(x,y) = x\mathbf{i} + y\mathbf{j} + (2a - x - y)\mathbf{k}, \qquad (x,y) \in \Omega := \{(x,y) : 0 \le x \le a,\ 0 \le y \le a\}.$$

Then $\mathbf{r}_x = \mathbf{i} - \mathbf{k}$ and $\mathbf{r}_y = \mathbf{j} - \mathbf{k}$, and thus

$$\mathbf{r}_x \times \mathbf{r}_y = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -1 \\ 0 & 1 & -1 \end{vmatrix} = \mathbf{i} + \mathbf{j} + \mathbf{k},$$

$$\begin{aligned}\mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y) &= (2xy\mathbf{i} + 2y(2a - x - y)\mathbf{j} + 2x(2a - x - y)\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k}) \\ &= 4ay - 2y^2 + 4ax - 2x^2 - 2xy.\end{aligned}$$

Hence,

$$\begin{aligned}\iint_S \mathbf{F}\cdot\mathbf{n}\,d\sigma &= \iint_\Omega \mathbf{F}(\mathbf{r}(x,y)) \cdot (\mathbf{r}_x \times \mathbf{r}_y)\,dx\,dy \\ &= \int_0^a \int_0^a (4ay - 2y^2 + 4ax - 2x^2 - 2xy)\,dy\,dx \\ &= \frac{13a^4}{6}.\end{aligned}$$

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