MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 8

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.4

5. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

   $\mathbf{F} = (x-y)\mathbf{i} + (y-x)\mathbf{j}$ $\quad$ C: The square bounded by $x = 0$, $x = 1$, $y = 0$, $y = 1$.

   Solution. For $M = x - y$, $N = y - x$, we have

   $$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = -1, \quad \frac{\partial N}{\partial x} = -1, \quad \frac{\partial N}{\partial y} = 1.$$

   By Green's Theorem,

   $$\text{Couterclockwise circulation} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_0^1 0\,dx\,dy = 0;$$

   $$\text{Outward flux} = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \int_0^1 \int_0^1 2\,dx\,dy = 2.$$

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8. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

   $\mathbf{F} = (x+y)\mathbf{i} - (x^2+y^2)\mathbf{j}$ $\quad$ C: The triangle bounded by $y = 0$, $x = 1$, $y = x$.

   Solution. For $M = x + y$, $N = -(x^2 + y^2)$, we have

   $$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 1, \quad \frac{\partial N}{\partial x} = -2x, \quad \frac{\partial N}{\partial y} = 2y.$$

   By Green's Theorem,

   $$\text{Couterclockwise circulation} = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy = \int_0^1 \int_0^x (-2x - 1)\,dy\,dx = -\frac{7}{6};$$

   $$\text{Outward flux} = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy = \int_0^1 \int_0^x (1 - 2y)\,dy\,dx = \frac{1}{6}.$$

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14. Use Green's Theorem to find the counterclockwise circulation and outward flux for the field $\mathbf{F}$ and curve C.

$\mathbf{F} = \left(\tan^{-1} \frac{y}{x}\right) \mathbf{i} + \ln(x^2 + y^2)\mathbf{j}$ C: The boundary of the region defined by the polar coordinate inequalities $1 \le r \le 2$, $0 \le \theta \le \pi$.

Solution. For $M = \tan^{-1} \frac{y}{x}$, $N = \ln(x^2 + y^2)$, we have

$$\frac{\partial M}{\partial x} = \frac{-y}{x^2+y^2}, \quad \frac{\partial M}{\partial y} = \frac{x}{x^2+y^2}, \quad \frac{\partial N}{\partial x} = \frac{2x}{x^2+y^2}, \quad \frac{\partial N}{\partial y} = \frac{2y}{x^2+y^2}.$$

By Green's Theorem,

$$\begin{aligned} \text{Couterclockwise circulation} &= \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \\ &= \iint_R \frac{x}{x^2+y^2}\,dx\,dy \\ &= \int_0^\pi \int_1^2 \frac{r\cos\theta}{r^2} \cdot r\,dr\,d\theta = \int_0^\pi \cos\theta\,d\theta = 0; \end{aligned}$$

$$\begin{aligned} \text{Outward flux} &= \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \\ &= \iint_R \frac{y}{x^2+y^2}\,dx\,dy \\ &= \int_0^\pi \int_1^2 \frac{r\sin\theta}{r^2} \cdot r\,dr\,d\theta = \int_0^\pi \sin\theta\,d\theta = 2. \end{aligned}$$

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16. Find the counterclockwise circulation and the outward flux of the field $\mathbf{F} = (-\sin y)\mathbf{i} + (x\cos y)\mathbf{j}$ around and over the square cut from the first quadrant by the lines $x = \pi/2$ and $y = \pi/2$.

Solution. For $M = -\sin y$, $N = x\cos y$, we have

$$\frac{\partial M}{\partial x} = 0, \quad \frac{\partial M}{\partial y} = -\cos y \quad \frac{\partial N}{\partial x} = \cos y, \quad \frac{\partial N}{\partial y} = -x\sin y.$$

By Green's Theorem,

$$\begin{aligned} \text{Couterclockwise circulation} &= \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy \\ &= \iint_R 2\cos y\,dx\,dy \\ &= \int_0^{\pi/2} \int_0^{\pi/2} 2\cos y\,dx\,dy \int_0^{\pi/2} \pi\cos y\,dy = \pi; \end{aligned}$$

$$\begin{aligned} \text{Outward flux} &= \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy \\ &= \iint_R (-x\sin y)\,dx\,dy \\ &= \int_0^{\pi/2} \int_0^{\pi/2} (-x\sin y)\,dx\,dy = \int_0^{\pi/2} (-\frac{\pi^2}{8}\sin y)\,dy = -\frac{\pi^2}{8}. \end{aligned}$$

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21. Apply Green's Theorem to evaluate the integral.

$\oint_C (y^2\,dx + x^2\,dy)$ $\quad$ C: The triangle bounded by $x = 0$, $x + y = 1$, $y = 0$.

Solution. For $M = y^2$, $N = x^2$, we have

$$\frac{\partial M}{\partial x} = 0, \quad \frac{\partial M}{\partial y} = 2y \quad \frac{\partial N}{\partial x} = 2x, \quad \frac{\partial N}{\partial y} = 0.$$

By Green's Theorem,

$$\begin{aligned}\oint_C (y^2\,dx + x^2\,dy) &= \iint_R (2x - 2y)\,dx\,dy \\ &= \int_0^1 \int_0^{1-x} (2x - 2y)\,dy\,dx \\ &= \int_0^1 (-3x^2 + 4x - 1)\,dx = 0.\end{aligned}$$

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23. Apply Green's Theorem to evaluate the integral.

$\oint_C (6y + x)\,dx + (y + 2x)\,dy$ $\quad$ C: The circle $(x-2)^2 + (y-3)^2 = 4$.

Solution. For $M = 6y + x$, $N = y + 2x$, we have

$$\frac{\partial M}{\partial x} = 1, \quad \frac{\partial M}{\partial y} = 6 \quad \frac{\partial N}{\partial x} = 2, \quad \frac{\partial N}{\partial y} = 1.$$

By Green's Theorem,

$$\oint_C (6y + x)\,dx + (y + 2x)\,dy = \iint_R (2 - 6)\,dx\,dy = -4 \cdot \text{ Area of the circle } = -16\pi.$$

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26. Use the Green's Theorem area formula to find the area of the region enclosed by the curve.
The circle $\mathbf{r}(t) = (a\cos t)\mathbf{i} + (b\sin t)\mathbf{j}, \quad 0 \le t \le 2\pi.$

Solution. By the Green's Theorem area formula,

$$\begin{aligned}\text{Area of } R &= \frac{1}{2}\oint_C x\,dy - y\,dx \\ &= \frac{1}{2}\int_0^{2\pi} [(a\cos t)(b\cos t) - (a\sin t)(b\sin t)]\,dt \\ &= \frac{1}{2}\int_0^{2\pi} ab\,dt = \pi ab.\end{aligned}$$

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31. Evaluate the integral

$$\oint_C 4x^3y\,dx + x^4\,dy$$

for any closed path C.

Solution. By Green's Theorem with $M = 4x^3y$, $N = x^4$, we have

$$\oint_C 4x^3y\,dx + x^4\,dy = \iint_R \left[\frac{\partial}{\partial x}(x^4) - \frac{\partial}{\partial y}(4x^3y)\right]\,dx\,dy$$
$$= \iint_R \left(4x^3 - 4x^3)\right)\,dx\,dy = 0.$$

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35. **Area and the centroid** Let A be the area and $\bar{x}$ the x-coordinate of the centroid of a region R that is bounded by a piecewise smooth, simple closed curve C in the xy-plane. Show that

$$\frac{1}{2}\oint_C x^2\,dy = -\oint_C xy\,dx = \frac{1}{3}\oint_C x^2\,dy - xy\,dx = A\bar{x}.$$

Solution. Let $\delta(x, y) = 1$. Then

$$\bar{x} = \frac{M_y}{M} = \frac{\iint_R x\delta(x,y)\,dA}{\iint_R \delta(x,y)\,dA} = \frac{\iint_R x\,dA}{\iint_R dA} = \frac{\iint_R x\,dA}{A}.$$

By Green's Theorem,

$$\frac{1}{2}\oint_C x^2\,dy = \frac{1}{2}\iint_R \frac{\partial}{\partial x}(x^2)]\,dx\,dy = \iint_R x\,dA = A\bar{x};$$

$$-\oint_C xy\,dx = -\iint_R \left[-\frac{\partial}{\partial y}(xy)\right]\,dx\,dy = \iint_R x\,dA = A\bar{x}.$$

and so

$$\frac{1}{3}\oint_C x^2\,dy - xy\,dx = \frac{2}{3}A\bar{x} + \frac{1}{3}A\bar{x} = A\bar{x}.$$

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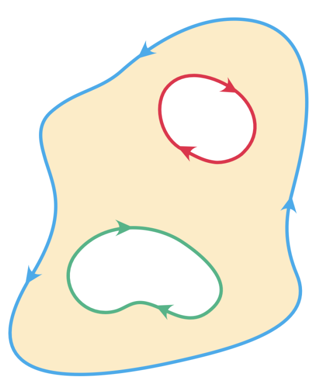
39. **Regions with many holes** Green's Theorem holds for a region R with any finite number of holes as long as the bounding curves are smooth, simple, and closed and we integrate over each component of the boundary in the direction that keeps R on our immediate left as we go along (see accompanying figure).



(a) Let $f(x, y) = \ln(x^2 + y^2)$ and let C be the circle $x^2 + y^2 = a^2$. Evaluate the flux integral

$$\oint_C \nabla f \cdot \mathbf{n}\,ds.$$

(b) Let K be an arbitrary smooth, simple closed curve in the plane that does not pass through $(0,0)$. Use Green's Theorem to show that

$$\oint_K \nabla f \cdot \mathbf{n}\, ds.$$

has two possible values, depending on whether $(0,0)$ lies inside K or outside K.

Solution. (a) Note that

$$\nabla f = M\mathbf{i} + N\mathbf{j} = \frac{2x}{x^2+y^2}\mathbf{i} + \frac{2y}{x^2+y^2}\mathbf{j}.$$

Since M, N are discontinuous at $(0,0)$, Green's Theorem is not applicable. We compute the integral directly. Let $x = a\cos t$, $y = a\sin t$. Then $dx = -a\sin t\, dt$, $dy = a\cos t\, dt$, $M = \frac{2}{a}\cos t$, $N = \frac{2}{a}\sin t$, $0 \le t \le 2\pi$. Hence,

$$\begin{aligned}\oint_C \nabla f \cdot \mathbf{n}\, ds &= \int_C M\, dy - N\, dx \\ &= \int_0^{2\pi} \left[(\frac{2}{a}\cos t)(a\cos t) - (\frac{2}{a}\sin t)(-a\sin t) \right] dt \\ &= \int_0^{2\pi} 2(\cos^2 t + \sin^2 t)\, dt = 4\pi.\end{aligned}$$

Note that this holds for any $a > 0$, so $\oint_C \nabla f \cdot \mathbf{n}\, ds = 4\pi$ for any circle C centered at $(0,0)$ traversed counterclockwise and $\oint_C \nabla f \cdot \mathbf{n}\, ds = -4\pi$ if C is traversed clockwise.

(b) If K does not enclose the point $(0,0)$, we may apply Green's Theorem:

$$\begin{aligned}\oint_K \nabla f \cdot \mathbf{n}\, ds = \int_C M\, dy - N\, dx &= \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy \\ &= \iint_R \left(\frac{2(y^2-x^2)}{(x^2+y^2)^2} + \frac{2(x^2-y^2)}{(x^2+y^2)^2} \right) dx\, dy \\ &= \iint_R 0\, dx\, dy = 0.\end{aligned}$$

If K does enclose the point $(0,0)$, we proceed as follows: Choose a small enough so that the circle C centered at $(0,0)$ of radius a lies entirely within K. Green's Theorem applies to the region R that lies between K and C. Thus, as before,

$$0 = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dx\, dy = \int_K M\, dy - N\, dx + \int_C M\, dy - N\, dx,$$

where K is traversed counterclockwise and C is traversed clockwise. Hence, by part (a),

$$\oint_K \nabla f \cdot \mathbf{n}\, ds = \int_K M\, dy - N\, dx = -\int_C M\, dy - N\, dx = 4\pi.$$

We have shown:

$$\oint_K \nabla f \cdot \mathbf{n}\, ds = \begin{cases} 0 & \text{if } (0,0) \text{ lies outside } K \\ 4\pi & \text{if } (0,0) \text{ lies inside } K. \end{cases}$$

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