MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 7

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.2

45. **Work and area** Suppose that $f(t)$ is differentiable and positive for $a \leq t \leq b$. Let C be the path $\mathbf{r}(t) = t\mathbf{i} + f(t)\mathbf{j}$, $a \leq t \leq b$, and $\mathbf{F} = y\mathbf{i}$. Is there an relation between the value of the work integral

$$\int_C \mathbf{F} \cdot d\mathbf{r}$$

and the area of the region bounded by the t-axis, the graph of f, and the lines $t = a$ and $t = b$? Give reasons for your answer.

Solution. Yes. The work and area have the same numerical value because

$$\begin{aligned}\int_C \mathbf{F} \cdot d\mathbf{r} &= \int_C y\mathbf{i}\, d\mathbf{r} \\ &= \int_a^b f(t)\mathbf{i} \cdot (\mathbf{i} + f'(t)\mathbf{j})\, dt \\ &= \int_a^b f(t)\, dt,\end{aligned}$$

which is the area under the curve because $f(t) > 0$. ◀

46. **Work done by a radial force with constant magnitude** A particle moves along the smooth curve $y = f(x)$ from $(a, f(a))$ to $(b, f(b))$. The force moving the particle has constant magnitude k and always points away from the origin. Show that the work done by the force is

$$\int_C \mathbf{F} \cdot \mathbf{T}\, ds = k[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}].$$

Solution. $C : \mathbf{r} = x\mathbf{i} + y\mathbf{j} = x\mathbf{i} + f(x)\mathbf{j} \implies \frac{d\mathbf{r}}{dx} = \mathbf{i} + f'(x)\mathbf{j}$; For a force with constant magnitude k and points away from the origin, $\mathbf{F} = \frac{k}{\sqrt{x^2+y^2}}(x\mathbf{i} + y\mathbf{j})$, and so

$$\mathbf{F} \cdot \frac{d\mathbf{r}}{dx} = \frac{kx}{\sqrt{x^2 + [f(x)]^2}} + \frac{kf(x)f'(x)}{\sqrt{x^2 + [f(x)]^2}} = k\frac{d}{dx}\sqrt{x^2 + [f(x)]^2} \quad \text{by the chain rule.}$$

Therefore,

$$\begin{aligned}\int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_C \mathbf{F} \cdot \frac{d\mathbf{r}}{dx}\, dx &= \int_a^b k\frac{d}{dx}\sqrt{x^2 + [f(x)]^2}\, dx \\ &= k\left[\sqrt{x^2 + [f(x)]^2}\right]_a^b \\ &= k[(b^2 + (f(b))^2)^{1/2} - (a^2 + (f(a))^2)^{1/2}].\end{aligned}$$

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Exercises 16.3

8. Find a potential function f for the field $\mathbf{F} = (y+z)\mathbf{i} + (x+z)\mathbf{j} + (x+y)\mathbf{k}$.

Solution. Suppose f is a potential function, that is $\nabla f = \mathbf{F}$. Then

$$\frac{\partial f}{\partial x} = y + z \implies f(x,y,z) = (y+z)x + g(y,z);$$

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = x + z \implies \frac{\partial g}{\partial y} = z \implies g(y,z) = zy + h(z);$$

$$\frac{\partial f}{\partial z} = x + y + h'(z) = x + y \implies h'(z) = 0 \implies h(z) = C.$$

Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = (y+z)x+zy+C$, where C is a constant. ◀

12. Find a potential function f for the field

$$\mathbf{F} = \frac{y}{1+x^2y^2}\mathbf{i} + \left(\frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}}\right)\mathbf{j} + \left(\frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z}\right)\mathbf{k}.$$

Solution. Suppose f is a potential function, that is $\nabla f = \mathbf{F}$. Then

$$\frac{\partial f}{\partial x} = \frac{y}{1+x^2y^2} \implies f(x,y,z) = \tan^{-1}(xy) + g(y,z);$$

$$\frac{\partial f}{\partial y} = \frac{x}{1+x^2y^2} + \frac{\partial g}{\partial y} = \frac{x}{1+x^2y^2} + \frac{z}{\sqrt{1-y^2z^2}} \implies \frac{\partial g}{\partial y} = \frac{z}{\sqrt{1-y^2z^2}}$$
$$\implies g(y,z) = \sin^{-1}(yz) + h(z);$$

$$\frac{\partial f}{\partial z} = \frac{y}{\sqrt{1-y^2z^2}} + h'(z) = \frac{y}{\sqrt{1-y^2z^2}} + \frac{1}{z} \implies h'(z) = \frac{1}{z} \implies h(z) = \ln|z| + C.$$

Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = \tan^{-1}(xy) + \sin^{-1}(yz) + \ln|z| + C$, where C is a constant. ◀

17. Show that the differential form in the integral is exact. Then evaluate the integral.

$$\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + \, dz.$$

Solution. Denote the differential from as $M\,dx + N\,dy + P\,dz$. Clearly, it is C^1 on $\mathbb{R}^3$, which is connected and simply connected. It is exact because

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \cos y \cos x = \frac{\partial M}{\partial y}.$$

Suppose $\nabla f = \sin y \cos x\mathbf{i} + \cos y \sin x\mathbf{j} + \mathbf{k}$. Then

$$\frac{\partial f}{\partial x} = \sin y \cos x \implies f(x,y,z) = \sin y \sin x + g(y,z);$$

$$\frac{\partial f}{\partial y} = \cos y \sin x + \frac{\partial g}{\partial y} = \cos y \sin x \implies \frac{\partial g}{\partial y} = 0 \implies g(y,z) = h(z);$$

$$\frac{\partial f}{\partial z} = h'(z) = 1 \implies h(z) = z + C.$$

Hence, $f(x, y, z) = \sin y \sin x + z + C$, where C is a constant. Therefore,

$$\int_{(1,0,0)}^{(0,1,1)} \sin y \cos x \, dx + \cos y \sin x \, dy + \, dz = f(0,1,1) - f(1,0,0) = 1 - 0 = 1.$$

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20. Find a potential function for the field and evaluate the integral as in Example 6.

$$\int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz) \, dx + \left(\frac{x^2}{y} - xz\right) \, dy - xy \, dz.$$

Solution. Denote the differential from as $M\,dx + N\,dy + P\,dz$. Clearly, it is C^1 on $\{(x, y, z) \in \mathbb{R}^3 : y > 0\}$, which is connected and simply connected. It is exact because

$$\frac{\partial P}{\partial y} = -x = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = -y = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = \frac{2x}{y} - z = \frac{\partial M}{\partial y}.$$

Suppose $\nabla f = (2x \ln y - yz)\mathbf{i} + \left(\frac{x^2}{y} - xz\right)\mathbf{j} - xy\mathbf{k}$. Then

$$\frac{\partial f}{\partial x} = 2x \ln y - yz \implies f(x, y, z) = x^2 \ln y - xyz + g(y, z);$$

$$\frac{\partial f}{\partial y} = \frac{x^2}{y} - xz + \frac{\partial g}{\partial y} = \frac{x^2}{y} - xz \implies \frac{\partial g}{\partial y} = 0 \implies g(y,z) = h(z);$$

$$\frac{\partial f}{\partial z} = -xy + h'(z) = -xy \implies h'(z) = 0 \implies h(z) = C.$$

Hence, $f(x, y, z) = x^2 \ln y - xyz + C$, where C is a constant. Therefore,

$$\begin{aligned}\int_{(1,2,1)}^{(2,1,1)} (2x \ln y - yz) \, dx + \left(\frac{x^2}{y} - xz\right) \, dy - xy \, dz &= f(2,1,1) - f(1,2,1) \\ &= (-2) - (\ln 2 - 2) = -\ln 2.\end{aligned}$$

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27. Find a potential function for $\mathbf{F}$.

$$\mathbf{F} = \frac{2x}{y}\mathbf{i} + \left(\frac{1 - x^2}{y^2}\right)\mathbf{j}, \qquad \{(x, y) : y > 0\}.$$

Solution. Note that $\mathbf{F}$ is a C^1 vector field on $\{(x, y) : y > 0\}$, which is connected and simply connected. Denote $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. It is conservative because

$$\frac{\partial N}{\partial x} = -\frac{2x}{y^2} = \frac{\partial M}{\partial y}.$$

So there is a potential function f such that $\nabla f = \mathbf{F}$. Then

$$\frac{\partial f}{\partial x} = \frac{2x}{y} \implies f(x,y) = \frac{x^2}{y} + g(y);$$

$$\frac{\partial f}{\partial y} = -\frac{x^2}{y^2} + g'(y) = \frac{1-x^2}{y^2} \implies g'(y) = \frac{1}{y^2} \implies g(y) = -\frac{1}{y} + C.$$

Hence, a potential function for $\mathbf{F}$ is $f(x,y) = \frac{x^2}{y} - \frac{1}{y} + C$, where C is a constant. ◀

29. **Work along different paths** Find the work done by $\mathbf{F} = (x^2+y)\mathbf{i} + (y^2+x)\mathbf{j} + ze^z\mathbf{k}$ over the following paths from $(1,0,0)$ to $(1,0,1)$.

   (a) The line segment $x = 1$, $y = 0$, $0 \le z \le 1$.

   (b) The helix $\mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j} + (t/2\pi)\mathbf{k}$, $0 \le t \le 2\pi$.

   (c) The x-axis from $(1,0,0)$ to $(0,0,0)$ followed by the parabola $z = x^2$, $y = 0$ from $(0,0,0)$ to $(1,0,1)$.

Solution. Note that $\mathbf{F}$ is a C^1 vector field on $\mathbb{R}^3$, which is connected and simply connected. Denote $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. It is conservative because

$$\frac{\partial P}{\partial y} = 0 = \frac{\partial N}{\partial z}, \quad \frac{\partial M}{\partial z} = 0 = \frac{\partial P}{\partial x}, \quad \frac{\partial N}{\partial x} = 1 = \frac{\partial M}{\partial y}.$$

So there is a potential function f such that $\nabla f = \mathbf{F}$. Then

$$\frac{\partial f}{\partial x} = x^2 + y \implies f(x,y,z) = \frac{1}{3}x^3 + xy + g(y,z);$$

$$\frac{\partial f}{\partial y} = x + \frac{\partial g}{\partial y} = y^2 + x \implies \frac{\partial g}{\partial y} = y^2 \implies g(y,z) = \frac{1}{3}y^3 + h(z);$$

$$\frac{\partial f}{\partial z} = h'(z) = ze^z \implies h(z) = ze^z - e^z + C.$$

Hence, a potential function for $\mathbf{F}$ is $f(x,y,z) = \frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z + C$, where C is a constant.

Since $\mathbf{F}$ is conservative, the work by $\mathbf{F}$ over each of the path is the same, and is given by

$$\text{Work} = \int_A^B \mathbf{F} \cdot d\mathbf{r} = \left[\frac{1}{3}x^3 + xy + \frac{1}{3}y^3 + ze^z - e^z\right]_{(1,0,0)}^{(1,0,1)} = \frac{1}{3} - (\frac{1}{3} - 1) = 1.$$

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33. (a) **Exact differential form** How are the constants a, b, and c related if the following differential form is exact?

$$(ay^2 + 2czx)\,dx + y(bx + cz)\,dy + (ay^2 + cx^2)\,dz$$

   (b) **Gradient field** For what values of b and c will

$$\mathbf{F} = (y^2 + 2czx)\mathbf{i} + y(bx + cz)\mathbf{j} + (y^2 + cx^2)\mathbf{k}$$

   be a gradient field?

Solution. (a) Denote the differential from as $M\,dx + N\,dy + P\,dz$. If it is exact, then

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \implies 2ay = cy,$$
$$\frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \implies 2cx = 2cx,$$
$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y} \implies by = 2ay.$$

So $b = c = 2a$.

(b) Note that $\mathbf{F}$ is a gradient field if and only if the differential form in (a) with $a = 1$ is exact. Since $\mathbf{F}$ is a C^1 vector field on $\mathbb{R}^3$, which is connected and simply connected, it is a gradient field when $b = c = 2$.

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38. **Gravitational field**

(a) Find a potential function for the gravitational field

$$\mathbf{F} = -GmM\frac{x\mathbf{i} + y\mathbf{j} + z\mathbf{k}}{(x^2 + y^2 + z^2)^{3/2}}$$

(G, m, and M are constants).

(b) Let P_1 and P_2 be points at distance s_1 and s_2 from the origin. Show that the work done by the gravitational field in part (a) in moving a particle from P_1 to P_2 is

$$GmM\left(\frac{1}{s_1} - \frac{1}{s_2}\right).$$

Solution. (a) Clearly $\frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$ is a C^1 vector field on $\mathbb{R}^3\backslash\{(0,0,0)\}$, which is connected and simply connected. Denote $M\mathbf{i}+N\mathbf{j}+P\mathbf{k} = \frac{x\mathbf{i}+y\mathbf{j}+z\mathbf{k}}{(x^2+y^2+z^2)^{3/2}}$. It is conservative because

$$\frac{\partial P}{\partial y} = \frac{-3yz}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial N}{\partial z},$$
$$\frac{\partial M}{\partial z} = \frac{-3zx}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial P}{\partial x},$$
$$\frac{\partial N}{\partial x} = \frac{-3xy}{(x^2 + y^2 + z^2)^{5/2}} = \frac{\partial M}{\partial y}.$$

So there is a potential function f such that $\nabla f = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$. Then

$$\frac{\partial f}{\partial x} = \frac{x}{(x^2 + y^2 + z^2)^{3/2}} \implies f(x, y, z) = -\frac{1}{(x^2 + y^2 + z^2)^{1/2}} + g(y, z);$$

$$\frac{\partial f}{\partial y} = \frac{y}{(x^2 + y^2 + z^2)^{3/2}} + \frac{\partial g}{\partial y} = \frac{y}{(x^2 + y^2 + z^2)^{3/2}} \implies \frac{\partial g}{\partial y} = 0 \implies g(y, z) = h(z);$$
$$\frac{\partial f}{\partial z} = \frac{z}{(x^2 + y^2 + z^2)^{3/2}} + h'(z) = \frac{z}{(x^2 + y^2 + z^2)^{3/2}} \implies h'(z) = 0 \implies h(z) = C.$$

Hence, $f(x, y, z) = -\frac{1}{(x^2 + y^2 + z^2)^{1/2}} + C$, where C is a constant.

Therefore, a potential function for $\mathbf{F}$ is $\frac{GmM}{(x^2 + y^2 + z^2)^{1/2}}$.

(b) If s is the distance of (x, y, z) from the origin, then $s = \sqrt{x^2 + y^2 + z^2}$. The work done by the gravitational field $\mathbf{F}$ is

$$\int_{P_1}^{P_2} \mathbf{F} \cdot d\mathbf{r} = \left[\frac{GmM}{(x^2 + y^2 + z^2)^{1/2}} \right]_{P_1}^{P_2} = \frac{GmM}{s_2} - \frac{GmM}{s_1} = GmM \left(\frac{1}{s_2} - \frac{1}{s_1} \right).$$

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