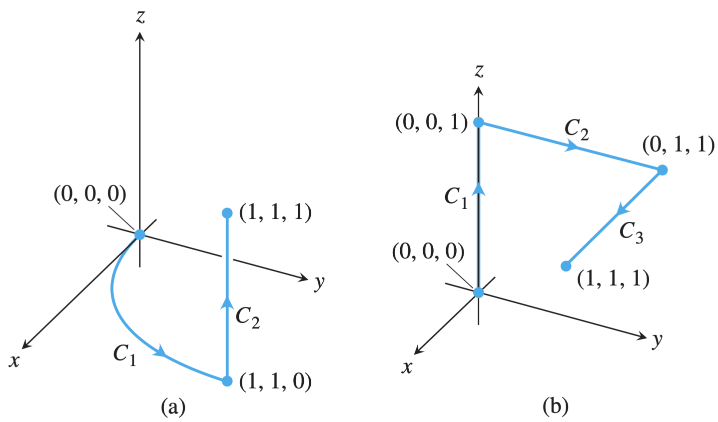
MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 6

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.1

15. Integrate $f(x, y, z) = x + \sqrt{y} - z^2$ over the path from $(0, 0, 0)$ to $(1, 1, 1)$ (see the accompanying figure) given by

$$C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad 0 \leq t \leq 1$$
$$C_2 : \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k}, \quad 0 \leq t \leq 1.$$



The paths of integration for Exercises 15 and 16.

Solution. $C_1 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j},\ 0 \leq t \leq 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + 4t^2}$; and $f(\mathbf{r}(t)) = t^2 + \sqrt{t^2} - 0 = t + |t| = 2t$ since $t \geq 0$. Hence

$$\int_{C_1} f(x, y, z)\, ds = \int_0^1 2t\sqrt{1 + 4t^2}\, dt = \left[\frac{1}{6}(1 + 4t^2)^{3/2}\right]_0^1 = \frac{1}{6}(5\sqrt{5} - 1).$$

$C_2 : \mathbf{r}(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k},\ 0 \leq t \leq 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{k} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$; and $f(\mathbf{r}(t)) = 1 + \sqrt{1} - t^2 = 2 - t^2$. So

$$\int_{C_2} f(x, y, z)\, ds = \int_0^1 (2 - t^2)(1)\, dt = \left[2t - \frac{t^3}{3}\right]_0^1 = \frac{5}{3}.$$

Therefore $\displaystyle\int_C f(x, y, z)\, ds = \int_{C_1} f(x, y, z)\, ds + \int_{C_2} f(x, y, z)\, ds = \frac{5}{6}\sqrt{5} + \frac{3}{2}.$ ◀

18. Integrate $f(x, y, z) = -\sqrt{x^2 + z^2}$ over the circle

$$\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k}, \quad 0 \leq t \leq 2\pi.$$

Solution. $\mathbf{r}(t) = (a\cos t)\mathbf{j} + (a\sin t)\mathbf{k},\ 0 \le t \le 2\pi. \implies \frac{d\mathbf{r}}{dt} = (-a\sin t)\mathbf{j} + (a\cos t)\mathbf{k} \implies \left|\frac{d\mathbf{r}}{dt}\right| = |a|$; and $f(\mathbf{r}(t)) = -\sqrt{0^2 + (a\sin t)^2} = \begin{cases} -|a|\sin t, & 0 \le t \le \pi \\ |a|\sin t, & \pi \le t \le 2\pi. \end{cases}$ Hence

$$\int_C f(x,y,z)\,ds = \int_0^{\pi} -|a|^2 \sin t\,dt + \int_{\pi}^{2\pi} |a|^2 \sin t\,dt = -4a^2.$$

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21. Find the line integral of $f(x,y) = ye^{x^2}$ along the curve $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j},\ -1 \le t \le 2$.

Solution. $\mathbf{r}(t) = 4t\mathbf{i} - 3t\mathbf{j},\ -1 \le t \le 2 \implies \frac{d\mathbf{r}}{dt} = 4\mathbf{i} - 3\mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 5$; and $f(\mathbf{r}(t)) = (-3t)e^{(4t)^2} = -3te^{16t^2}$. Hence

$$\int_C f(x,y,z)\,ds = \int_{-1}^{2} -3te^{16t^2} \cdot 5\,dt = \left[-\frac{15}{32}e^{16t^2}\right]_{-1}^{2} = \frac{15}{32}(e^{16} - e^{64}).$$

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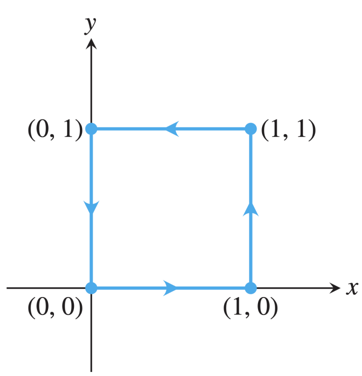
26. Evaluate $\displaystyle\int_C \frac{1}{x^2+y^2+1}\,ds$ where C is given in the accompanying figure.



Solution. $C_1 : \mathbf{r}(t) = t\mathbf{i},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$;
$C_2 : \mathbf{r}(t) = \mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$;
$C_3 : \mathbf{r}(t) = (1-t)\mathbf{i} + \mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = -\mathbf{i} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$;
$C_4 : \mathbf{r}(t) = (1-t)\mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = -\mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = 1$.

Hence,

$$\begin{aligned}\int_C f(x,y,z)\,ds &= \int_{C_1} f(x,y,z)\,ds + \int_{C_2} f(x,y,z)\,ds + \int_{C_3} f(x,y,z)\,ds + \int_{C_4} f(x,y,z)\,ds \\ &= \int_0^1 \frac{dt}{t^2+1} + \int_0^1 \frac{dt}{t^2+2} + \int_0^1 \frac{dt}{(1-t)^2+2} + \int_0^1 \frac{dt}{(1-t)^2+1} \\ &= \left[\tan^{-1} t\right]_0^1 + \frac{1}{\sqrt{2}}\left[\tan^{-1}(\frac{t}{\sqrt{2}})\right]_0^1 + \frac{1}{\sqrt{2}}\left[\tan^{-1}(\frac{t-1}{\sqrt{2}})\right]_0^1 + \left[-\tan^{-1}(1-t)\right]_0^1 \\ &= \frac{\pi}{2} + \sqrt{2}\tan^{-1}(\frac{1}{\sqrt{2}}).\end{aligned}$$

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28. Integrate f over the given curve.
$f(x, y) = (x + y^2)/\sqrt{1 + x^2}$, $\quad C : y = x^2/2$ from $(1, 1/2)$ to $(0, 0)$.

Solution. $C : \mathbf{r}(t) = (1 - t)\mathbf{i} + \frac{1}{2}(1 - t)^2\mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = -\mathbf{i} - (1 - t)\mathbf{j} \implies \left|\frac{d\mathbf{r}}{dt}\right| = \sqrt{1 + (1 - t)^2}$. Hence

$$\begin{aligned}\int_C f(x, y, z)\, ds &= \int_0^1 \frac{(1 - t) + \frac{1}{4}(1 - t)^4}{\sqrt{1 + (1 - t)^2}} \cdot \sqrt{1 + (1 - t)^2}\, dt \\ &= \int_0^1 \left((1 - t) + \frac{1}{4}(1 - t)^4\right) dt \\ &= \left[-\frac{1}{2}(1 - t)^2 - \frac{1}{20}(1 - t)^5\right]_0^1 = \frac{11}{20}.\end{aligned}$$

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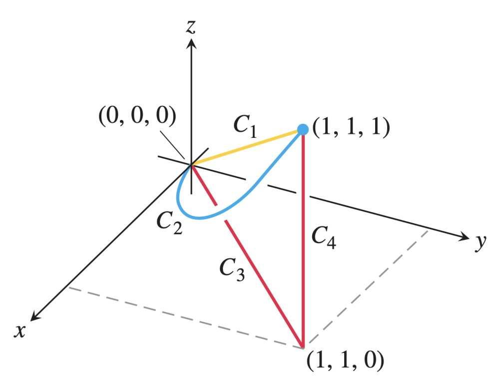
Exercises 16.2

9. Find the line integral of $\mathbf{F} = \sqrt{z}\mathbf{i} - 2x\mathbf{j} + \sqrt{y}\mathbf{k}$ from $(0, 0, 0)$ to $(1, 1, 1)$ over each of the following paths in the accompanying figure.

(a) The straight-line path $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 1$.
(b) The curved path $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k}, \quad 0 \le t \le 1$.
(c) The path $C_3 \cup C_4$ consisting of the line segment from $(0, 0, 0)$ to $(1, 1, 0)$ followed by the segment from $(1, 1, 0)$ to $(1, 1, 1)$.



Solution. (a) $C_1 : \mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} + \mathbf{j} + \mathbf{k}$. Hence,

$$\begin{aligned}\int_{C_1} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 F(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 (\sqrt{t}\mathbf{i} - 2t\mathbf{j} + \sqrt{t}\mathbf{k}) \cdot (\mathbf{i} + \mathbf{j} + \mathbf{k})\, dt \\ &= \int_0^1 (2\sqrt{t} - 2t)\, dt = \left[\frac{4}{3}t^{3/2} - t^2\right]_0^1 = \frac{1}{3}.\end{aligned}$$

(b) $C_2 : \mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^4\mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}}{dt} = \mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k}$. Hence,

$$\begin{aligned}\int_{C_2} \mathbf{F} \cdot d\mathbf{r} &= \int_0^1 F(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt = \int_0^1 (t^2\mathbf{i} - 2t\mathbf{j} + t\mathbf{k}) \cdot (\mathbf{i} + 2t\mathbf{j} + 4t^3\mathbf{k})\, dt \\ &= \int_0^1 (4t^4 - 3t^2)\, dt = \left[\frac{4}{5}t^{3/2} - t^3\right]_0^1 = -\frac{1}{5}.\end{aligned}$$

(c) $C_3 : \mathbf{r}_3(t) = t\mathbf{i} + t\mathbf{j},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}_3}{dt} = \mathbf{i} + \mathbf{j} \implies \mathbf{F}(\mathbf{r}_3(t)) \cdot \frac{d\mathbf{r}_3}{dt} = -2t;$
$C_4 : \mathbf{r}_4(t) = \mathbf{i} + \mathbf{j} + t\mathbf{k},\ 0 \le t \le 1 \implies \frac{d\mathbf{r}_4}{dt} = \mathbf{k} \implies \mathbf{F}(\mathbf{r}_4(t)) \cdot \frac{d\mathbf{r}_4}{dt} = 1.$ Hence,

$$\int_{C_3 \cup C_4} \mathbf{F} \cdot d\mathbf{r} = \int_{C_3} \mathbf{F} \cdot d\mathbf{r} + \int_{C_4} \mathbf{F} \cdot d\mathbf{r} = \int_0^1 -2t\,dt + \int_0^1 1\,dt = -1 + 1 = 0.$$

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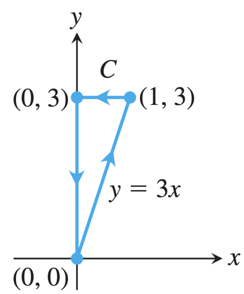
16. $\int_C \sqrt{x+y}\,dx$, where C is given in the accompanying figure.



Solution. $C_1 : x = t, y = 3t,\ 0 \le t \le 1 \implies dx = dt;$
$C_2 : x = (1-t), y = 3,\ 0 \le t \le 1 \implies dx = -dt;$
$C_3 : x = 0, y = 3 - t\ 0 \le t \le 3 \implies dx = 0.$ Hence,

$$\begin{aligned}\int_C \sqrt{x+y}\,dx &= \int_{C_1} \sqrt{x+y}\,dx + \int_{C_2} \sqrt{x+y}\,dx + \int_{C_3} \sqrt{x+y}\,dx \\ &= \int_0^1 \sqrt{t+3t}\,dt + \int_0^1 \sqrt{(1-t)+3}\,(-1)dt + \int_0^3 \sqrt{0+(3-t)} \cdot 0 \\ &= \int_0^1 2\sqrt{t}\,dt - \int_0^1 \sqrt{4-t}\,dt \\ &= \left[\frac{4}{3}t^{3/2}\right]_0^1 + \left[\frac{2}{3}(4-t)^{3/2}\right]_0^1 = 2\sqrt{3} - 4.\end{aligned}$$

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21. Find the work done by $\mathbf{F}$ over the curve in the direction of increasing t.

$\mathbf{F} = z\mathbf{i} + x\mathbf{j} + y\mathbf{k}$
$\mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 2\pi.$

Solution. $C : \mathbf{r}(t) = (\sin t)\mathbf{i} + (\cos t)\mathbf{j} + t\mathbf{k},\ 0 \le t \le 2\pi \implies \frac{d\mathbf{r}}{dt} = (\cos t)\mathbf{i} + (-\sin t)\mathbf{j} + \mathbf{k};$
and $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = (t\mathbf{i} + (\sin t)\mathbf{j} + (\cos t)\mathbf{k}) \cdot ((\cos t)\mathbf{i} + (-\sin t)\mathbf{j} + \mathbf{k}) = t\cos t - \sin^2 t + \cos t.$
Hence,

$$\begin{aligned}\text{Work done by } \mathbf{F} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} (t\cos t - \sin^2 t + \cos t)\,dt \\ &= \left[\cos t + t\sin t - \frac{t}{2} + \frac{\sin 2t}{4} + \sin t\right]_0^{2\pi} = -\pi.\end{aligned}$$

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26. Evaluate $\int_C \mathbf{F} \cdot d\mathbf{r}$ for the vector field $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$ counterclockwise along the unit circle $x^2 + y^2 = 1$ from $(1, 0)$ to $(0, 1)$.

Solution. $C : \mathbf{r}(t) = (\cos t)\mathbf{i} + (\sin t)\mathbf{j},\ 0 \le t \le \pi/2 \implies \frac{d\mathbf{r}}{dt} = (-\sin t)\mathbf{i} + (\cos t)\mathbf{j}$; and $\mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt} = ((\sin t)\mathbf{i} - (\cos t)\mathbf{j}) \cdot ((-\sin t)\mathbf{i} + (\cos t)\mathbf{j}) = -\sin^2 t - \cos^2 t = -1$. Hence,

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{\pi/2} (-1)\, dt = -\frac{\pi}{2}.$$

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32. Find the circulation and flux of the field $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j}$ around and across the closed semicircular path that consists of the semicircular arch $\mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j}$, $0 \le t \le \pi$, followed by the line segment $\mathbf{r}_2(t) = t\mathbf{i}$, $-a \le t \le a$.

Solution. $C_1 : \mathbf{r}_1(t) = (a\cos t)\mathbf{i} + (a\sin t)\mathbf{j},\ 0 \le t \le \pi \implies \frac{d\mathbf{r}_1}{dt} = (-a\sin t)\mathbf{i} + (a\cos t)\mathbf{j}$;
$\mathbf{F} \cdot \frac{d\mathbf{r}_1}{dt} = -a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t$ and
$M\,dy - N\,dx = (a\cos t)^2(a\cos t)\,dt - (a\sin t)^2(-a\sin t)\,dt = a^3(\cos^3 t + \sin^3 t)\,dt.$

$C_2 : \mathbf{r}_2(t) = t\mathbf{i},\ -a \le t \le a \implies \frac{d\mathbf{r}_2}{dt} = \mathbf{i}$;
$\mathbf{F} \cdot \frac{d\mathbf{r}_2}{dt} = t^2$ and
$M\,dy - N\,dx = t^2 \cdot 0 - 0^2 \cdot dt = 0.$

Hence,

$$\begin{aligned}
\text{Circulation} &= \int_C \mathbf{F} \cdot d\mathbf{r} = \int_{C_1} \mathbf{F} \cdot d\mathbf{r} + \int_{C_2} \mathbf{F} \cdot d\mathbf{r} \\
&= \int_0^{\pi} (-a^3 \sin t \cos^2 t + a^3 \cos t \sin^2 t)\,dt + \int_{-a}^{a} t^2\,dt \\
&= \left[\frac{a^3}{3}\cos^3 t + \frac{a^3}{3}\sin^3 t\right]_0^{\pi} + \frac{2a^3}{3} \\
&= 0;
\end{aligned}$$

$$\begin{aligned}
\text{Flux} &= \oint_C M\,dy - N\,dx = \oint_{C_1} M\,dy - N\,dx + \oint_{C_2} M\,dy - N\,dx \\
&= \int_0^{\pi} a^3(\cos^3 t + \sin^3 t)\,dt + \int_{-a}^{a} 0 \\
&= 2a^3 \int_0^{\pi} (1 - \cos^2 t)\sin t\,dt \\
&= -a^3\left[\cos t - \frac{\cos^3 t}{3}\right]_0^{\pi} = \frac{4a^3}{3}.
\end{aligned}$$

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