MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 5

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Practice Exercises 15

15. **Volume of the region under a paraboloid** Find the volume under the paraboloid $z = x^2 + y^2$ above the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.

Solution. Volume $= \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx = \int_0^1 \left[2x^2 + \frac{(2-x)^3}{3} - \frac{7x^3}{3}\right] dx = \frac{4}{3}$. ◀

17. Find the average value of $f(x, y) = xy$ over the square bounded by the lines $x = 1$, $y = 1$ in the first quadrant.

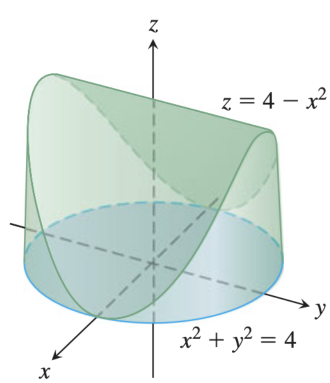
Solution. Average $= \frac{1}{1^2} \int_0^1 \int_0^1 xy\, dx\, dy = \left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{4}$. ◀

20. Evaluate the integral by changing to polar coordinate: $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\, dx\, dy$.

Solution. $\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \ln(x^2 + y^2 + 1)\, dx\, dy = \int_0^{2\pi} \int_0^1 \ln(r^2 + 1)\, r\, dr\, d\theta$

$= \int_0^{2\pi} \left[\frac{1}{2}(u \ln u - u)\right]_1^2 d\theta = \pi(2 \ln 2 - 1)$. ◀

28. **Volume** Find the volume of the solid that is bounded above by the cylinder $z = 4 - x^2$, on the sides by the cylinder $x^2 + y^2 = 4$, and below by the xy-plane.



Solution. Volume $= \int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_0^{4-x^2} dz\, dy\, dx = 4 \int_0^2 \int_0^{\sqrt{4-x^2}} (4 - x^2)\, dy\, dx$

$= 4 \int_0^2 (4 - x^2)^{3/2}\, dx = \left[x(4 - x^2)^{3/2} + 6x\sqrt{4 - x^2} + 24 \sin^{-1} \frac{x}{2}\right]_0^2 = 24 \sin^{-1} 1 = 12\pi$. ◀

31. **Cylindrical to rectangular coordinates** Convert

$$\int_0^{2\pi}\int_0^{\sqrt{2}}\int_r^{\sqrt{4-r^2}} 3\,dz\,r dr\,d\theta, \quad r \geq 0$$

to **(a)** rectangular coordinates with the order of integration $dz\,dx\,dy$ and **(b)** spherical coordinates. Then **(c)** evaluate one of the integrals.

Solution. (a) $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}}\int_{-\sqrt{2-y^2}}^{\sqrt{2-y^2}}\int_{\sqrt{x^2+y^2}}^{\sqrt{4-x^2-y^2}} 3\,dz\,dx\,dy.$

(b) $\displaystyle = \int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$

(c) Using (b), $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{2} 3\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\left[-\cos\phi\right]_0^{\pi/4}\left[\rho^3\right]_0^2 = 2\pi(8-4\sqrt{2}).$

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33. **Rectangular to spherical coordinates (a)** Convert to spherical coordinates. Then **(b)** evaluate the new integral.

$$\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_{\sqrt{x^2+y^2}}^{1} dz\,dy\,dx.$$

Solution. (a) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$

(b) $\displaystyle\int_0^{2\pi}\int_0^{\pi/4}\int_0^{\sec\phi} \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/4} \frac{1}{3}\sec^2\phi\tan\phi\,d\rho\,d\phi\,d\theta$

$\displaystyle = \frac{2\pi}{3}\left[\frac{\tan^2\phi}{2}\right]_0^{\pi/4} = \frac{\pi}{3}.$

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34. **Rectangular, cylindrical, and spherical coordinates** Write an iterated triple integral for the integral of $f(x,y,z) = 6+4y$ over the region in the first octant bounded by the cone $z = \sqrt{x^2+y^2}$, the cylinder $x^2+y^2 = 1$, and the coordinate planes in **(a)** rectangular coordinates, **(b)** cylindrical coordinates, and **(c)** spherical coordinates. Then **(d)** find the integral of f by evaluating one of the triple integrals.

Solution. (a) $\displaystyle\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{x^2+y^2}} (6+4y)\,dz\,dy\,dx.$

(b) $\displaystyle\int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\,dz\,r dr\,d\theta.$

(c) $\displaystyle\int_0^{\pi/2}\int_{\pi/4}^{\pi/2}\int_0^{\csc\phi} (6+4\rho\sin\theta\sin\phi)(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta.$

(d) Using (b), $\displaystyle\int_0^{\pi/2}\int_0^1\int_0^r (6+4r\sin\theta)\,dz\,r dr\,d\theta = \int_0^{\pi/2}\int_0^1 (6r^2+4r^3\sin\theta)dr\,d\theta$

$\displaystyle = \int_0^{\pi/2} (2+\sin\theta)\,d\theta = \left[2\theta-\cos\theta\right]_0^{\pi/2} = \pi+1.$

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36. **Rectangular to cylindrical coordinates** The volume of a solid is

$$\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx.$$

(a) Describe the solid by giving equations for the surfaces that form its boundary.

(b) Convert the integral to cylindrical coordinates but do not evaluate the integral.

Solution. (a) The solid is bounded on the top and bottom by the sphere $x^2+y^2+z^2=4$, on the right by the circular cylinder $(x-1)^2+y^2=1$, on the left by the plane $y=0$.

(b) $\displaystyle\int_0^2 \int_0^{\sqrt{2x-x^2}} \int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} dz\,dy\,dx = \int_0^{\pi/2} \int_0^{2\cos\theta} \int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} dz\,r dr\,d\theta.$

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37. **Spherical versus cylindrical coordinates** Triple integrals involving spherical shapes do not always require spherical coordinates for convenient evaluation. Some calculations may be accomplished more easily with cylindrical coordinates. As a case in point, find the volume of the region bounded above by the sphere $x^2+y^2+z^2=8$ and below by the plane $z=2$ by using **(a)** cylindrical coordinates and **(b)** spherical coordinates.

Solution. (a) Volume $\displaystyle = \int_0^{2\pi} \int_0^2 \int_2^{\sqrt{8-r^2}} dz\,r dr\,d\theta = 2\pi \int_0^2 (r\sqrt{8-r^2}-2r)\,dr$

$$= 2\pi \left[-\frac{1}{3}(8-r^2)^{3/2} - r^2\right]_0^2 = \frac{8\pi(4\sqrt{2}-5)}{3}.$$

(b) Volume $\displaystyle = \int_0^{2\pi} \int_0^{\pi/4} \int_{2\sec\phi}^{2\sqrt{2}} \rho^2 \sin\phi\,d\rho\,d\phi\,d\theta = \frac{8}{3}\int_0^{2\pi} \int_0^{\pi/4} \left(2\sqrt{2}\sin\phi - \sec^3\phi\sin\phi\right) d\phi\,d\theta$

$$= \frac{32\sqrt{2}\pi}{3}\int_0^{\pi/4} \sin\phi\,d\phi - \frac{16\pi}{3}\int_0^{\pi/4} \sec^2\phi\tan\phi\,d\phi = \frac{32\sqrt{2}\pi}{3}(1-\frac{1}{\sqrt{2}}) - \frac{8\pi}{3} = \frac{8\pi(4\sqrt{2}-5)}{3}.$$

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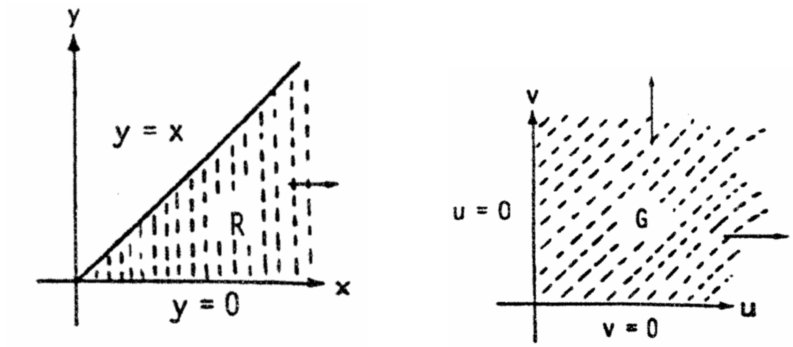
53. Show that if $u=x-y$ and $v=y$, then for any continuous f

$$\int_0^\infty \int_0^x e^{-sx} f(x-y,y)\,dy\,dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u,v)\,du\,dv.$$

Solution. We have $x=u+v$ and $y=v$, and thus $J(u,v) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1.$

The boundary of the image G is obtained from the boundary of R as follows:

xy-equations for the boundary of R	Corresponding uv-equations for the boundary of G
$y=x$	$v=u+v$, i.e., $u=0$
$y=0$	$v=0$



So,

$$\int_0^\infty \int_0^x e^{-sx} f(x-y,y)\,dy\,dx = \int_0^\infty \int_0^\infty e^{-s(u+v)} f(u,v)\,du\,dv.$$

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