MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 4

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 15.6

2. **Find moments of inertia** Find the moments of inertia about the coordinate axes of a thin rectangular plate of constant density δ bounded by the lines $x = 3$ and $y = 3$ in the first quadrant.

   Solution. $\mathbf{I}_x = \iint_R y^2 \delta \, dA = \delta \int_0^3 \int_0^3 y^2 \, dy \, dx = 27\delta$;

   $\mathbf{I}_y = \iint_R x^2 \delta \, dA = \delta \int_0^3 \int_0^3 x^2 \, dy \, dx = 27\delta.$ ◀

3. **Finding a centroid** Find the centroid of the region in the first quadrant bounded by the x-axis, the parabola $y^2 = 2x$, and the line $x + y = 4$.

   Solution. $\mathbf{M} = \int_0^2 \int_{y^2/2}^{4-y} dx \, dy = \int_0^2 \left(4 - y - \frac{y^2}{2}\right) dy = \frac{14}{3}$;

   $\mathbf{M}_y = \int_0^2 \int_{y^2/2}^{4-y} x \, dx \, dy = \frac{1}{2} \int_0^2 \left(16 - 8y + y^2 - \frac{y^4}{4}\right) dy = \frac{128}{15}$;

   $\mathbf{M}_x = \int_0^2 \int_{y^2/2}^{4-y} y \, dx \, dy = \int_0^2 \left(4y - y^2 - \frac{y^3}{2}\right) dy = \frac{10}{3}.$

   Hence, the centroid is $(\bar{x}, \bar{y}) = (\mathbf{M}_y/\mathbf{M}, \mathbf{M}_x/\mathbf{M}) = (\frac{64}{35}, \frac{5}{7})$. ◀

Exercises 15.8

3. (a) Solve the system

   $$u = 3x + 2y, \quad v = x + 4y$$

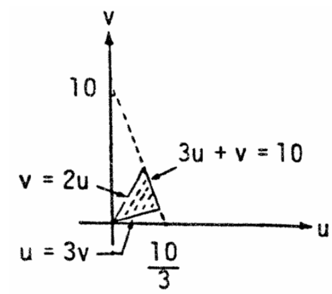
   for x and y in terms of u and v. Then find the value of the Jacobian $\partial(x, y)/\partial(u, v)$.

   (b) Find the image under the transformation $u = 3x + 2y$, $v = x + 4y$ of the triangular region in the xy-plane bounded by the x-axis, the y-axis and the line $x + y = 1$. Sketch the transformed region in the uv-plane.

   Solution. (a) $x = \frac{1}{5}(2u - v)$ and $y = \frac{1}{10}(-u + 3v)$. $\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \frac{2}{5} & -\frac{1}{5} \\ -\frac{1}{10} & \frac{3}{10} \end{vmatrix} = \frac{6}{50} - \frac{1}{50} = \frac{1}{10}.$

   (b) $y = 0 \implies u = 3v; \quad x = 0 \implies v = 2u;$

   $x + y = 1 \implies \frac{1}{5}(2u - v) + \frac{1}{10}(3v - u) = 1 \implies 3u + v = 10.$



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7. Use the transformation in Exercise 3 to evaluate the integral

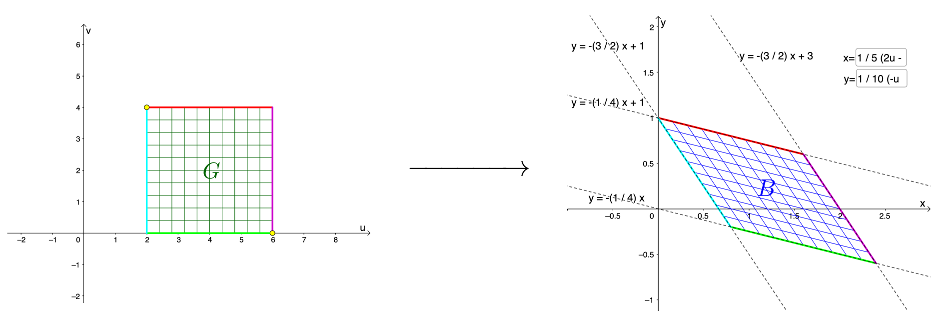
$$\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy$$

for the region R in the first quadrant bounded by the lines $y = -(3/2)x+1$, $y = -(3/2)x+3$, $y = -(1/4)x$, and $y = -(1/4)x + 1$.

Remark: The requirement for R lying in the first quadrant is a typo.

Solution. Let G be the image of R under the transformation $u = 3x + 2y$, $v = x + 4y$.

xy-equations for the boundary of R	corresponding uv-equations for the boundary of G
$y = -\frac{3}{2}x + 1$	$u = 2$
$y = -\frac{3}{2}x + 3$	$u = 6$
$y = -\frac{1}{4}x$	$v = 0$
$y = -\frac{1}{4}x + 1$	$v = 4$



Hence,

$$\begin{aligned}
\iint_R (3x^2 + 14xy + 8y^2)\, dx\, dy &= \iint_R (3x + 2y)(x + 4y)\, dx\, dy \\
&= \iint_G uv \left| \frac{\partial(x, y)}{\partial(u, v)} \right| du\, dv \\
&= \frac{1}{10} \iint_G uv\, du\, dv \\
&= \frac{1}{10} \int_0^4 \int_2^6 uv\, du\, dv \\
&= \frac{1}{10} \left[\frac{u^2}{2} \right]_2^6 \left[\frac{v^2}{2} \right]_0^4 = \frac{64}{5}.
\end{aligned}$$

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12. **The area of an ellipse** The area πab of the ellipse $x^2/a^2 + y^2/b^2 = 1$ can be found by integrating the function $f(x,y) = 1$ over the region bounded by the ellipse in the xy-plane. Evaluating the integral directly requires a trigonometric substitution. An easier way to evaluate the integral is to use the transformation $x = au$, $y = bv$ and evaluate the transformed integral over the disk G: $u^2 + v^2 \leq 1$ in the uv-plane. Find the area this way.

Solution. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix} = ab$. Hence,

$$\begin{aligned}\text{Area} &= \iint_R dy\,dx = \iint_G ab\,du\,dv = \int_{-1}^{1}\int_{-\sqrt{1-v^2}}^{\sqrt{1-v^2}} ab\,du\,dv \\ &= 2ab\int_{-1}^{1}\sqrt{1-v^2}\,dv = 2ab\left[\frac{v}{2}\sqrt{1-v^2} + \frac{1}{2}\arcsin u\right]_{-1}^{1} = \pi ab.\end{aligned}$$

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16. Use the transformation $x = u^2 - v^2$, $y = 2uv$ to evaluate the integral

$$\int_0^1\int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\,dy\,dx$$

(*Hint:* Show that the image of the triangular region G with vertices $(0,0)$, $(1,0)$, $(1,1)$ in the uv-plane is the region of integration R in the xy-plane defined by the limits of integration.)

Solution. For $x = u^2 - v^2$, $y = 2uv$, $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2u & -2v \\ 2v & 2u \end{vmatrix} = 4(u^2+v^2)$;

$$\begin{aligned} v = 0 &\implies y = 0; \\ u = v &\implies x = 0; \\ u = 1 &\implies y^2 = (2v)^2 = 4(1-(1-v^2)) = 4(1-x). \end{aligned}$$

Hence, the transformation maps the triangular region G with vertices $(0,0)$, $(1,0)$, $(1,1)$ onto the region R. Therefore,

$$\begin{aligned}\int_0^1\int_0^{2\sqrt{1-x}} \sqrt{x^2+y^2}\,dy\,dx &= \int_0^1\int_0^u \sqrt{(u^2-v^2)^2+(2uv)^2}\cdot 4(u^2+v^2)\,dv\,du \\ &= 4\int_0^1\int_0^u (u^2+v^2)^2\,dv\,du = 4\int_0^1 \left[u^4v + \frac{2}{3}u^2v^3 + \frac{1}{5}v^5\right]_0^u du \\ &= \frac{112}{15}\int_0^1 u^5\,du = \frac{56}{45}.\end{aligned}$$

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18. **Volume of an ellipsoid** Find the volume of the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

(*Hint:* Let $x = au$, $y = bv$, and $z = cw$. Then find the volume of an appropriate region in uvw-space.)

Solution. The transformation $x = au$, $y = bv$, $z = cw$ satisfies

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc,$$

and takes the ellipsoid region $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} \leq 1$ in xyz-space into the spherical region $u^2 + w^2 + z^2 \leq 1$ in uvw-space (which has volume $\frac{4}{3}\pi$.)

Hence, the volume of the ellipsoid is

$$V = \iiint_R dx\, dy\, dz = \iiint_G abc\, du\, dv\, dw = \frac{4\pi abc}{3}.$$

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20. Let D be the region in xyz-plane defined by the inequalities

$$1 \leq x \leq 2, \quad 0 \leq xy \leq 2, \quad 0 \leq z \leq 1.$$

Evaluate

$$\iiint_D (x^2y + 3xyz)\, dx\, dy\, dz$$

by applying the transformation

$$u = x, \quad v = xy, \quad w = 3z$$

and integrating over an appropriate region G in uvw-space.

Solution. For the transformation $u = x$, $v = xy$, $w = 3z$, we have

$$x = u, \quad y = \frac{v}{u}, \quad z = \frac{1}{3}w,$$

and

$$J(u,v,w) = \begin{vmatrix} 1 & 0 & 0 \\ -\frac{v}{u^2} & \frac{1}{u} & 0 \\ 0 & 0 & \frac{1}{3} \end{vmatrix} = \frac{1}{3u}.$$

Hence

$$\begin{aligned} \iiint_D (x^2y + 3xyz)\, dx\, dy\, dz &= \iiint_G \left[u^2(\frac{v}{u}) + 2u(\frac{v}{u})(\frac{w}{3})\right] |J(u,v,w)|\, du\, dv\, dw \\ &= \frac{1}{3}\int_0^3 \int_0^2 \int_1^2 (v + \frac{vw}{u})\, du\, dv\, dw \\ &= \frac{1}{3}\int_0^3 \int_0^2 (v + vw \ln 2)\, dv\, dw \\ &= \frac{2}{3}(3 + \frac{9}{2}\ln 2) = 2 + 3\ln 2. \end{aligned}$$

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27. **Inverse transform** The equations $x = g(u,v)$, $y = h(u,v)$ in Figure 15.54 transform the region G in the uv-plane into the region R in the xy-plane. Since the substitution transformation is one-to-one with continuous first partial derivatives, it has an inverse transformation and there are equations $u = \alpha(x,y)$, $v = \beta(x,y)$ with continuous first partial derivatives transforming R back to G. Moreover, the Jacobian determinants of the transformations are related reciprocally by

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} \tag{1}$$

Equation (1) is proved in advanced calculus. Use it to find the area of the region R in the first quadrant of the xy-plane bounded by the lines $y = 2x$, $2y = x$, and the curves $xy = 2$, $2xy = 1$ for $u = xy$ and $v = y/x$.

Solution. For $u = xy$ and $v = y/x$,

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} y & x \\ -\frac{y}{x^2} & \frac{1}{x} \end{vmatrix} = \frac{2y}{x},$$

and hence

$$\frac{\partial(x,y)}{\partial(u,v)} = \left(\frac{\partial(u,v)}{\partial(x,y)}\right)^{-1} = \frac{x}{2y} = \frac{1}{2v}.$$

Moreover, G is the region in the uv-plane bounded by the lines $u = \frac{1}{2}$, $u = 2$, $v = \frac{1}{2}$ and $v = 2$. Therefore,

$$\begin{aligned} \text{Area of } R &= \int_R dx\,dy = \int_G \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv \\ &= \int_{1/2}^{2}\int_{1/2}^{2} \frac{1}{2v}\,du\,dv \\ &= \frac{1}{2}(2-\frac{1}{2})(\ln 2 - \ln\frac{1}{2}) = \frac{3}{2}\ln 2. \end{aligned}$$

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28. *(Continuation of Exercise 27.)* For the region R described in Exercise 27, evaluate the integral $\iint_R y^2\,dA$.

Solution. By Exercise 27,

$$\begin{aligned} \iint_R y^2\,dA &= \int_{1/2}^{2}\int_{1/2}^{2} (uv)\cdot\frac{1}{2v}\,du\,dv \\ &= \frac{1}{2}\int_{1/2}^{2}\int_{1/2}^{2} u\,du\,dv \\ &= \frac{1}{2}(2-\frac{1}{2})(\frac{2^2}{2} - \frac{(1/2)^2}{2}) = \frac{45}{32}. \end{aligned}$$

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