MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 3

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

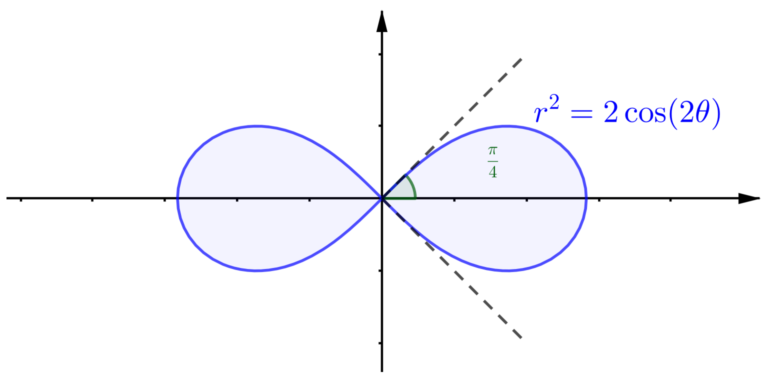
Exercises 15.4

34. **Average height of a cone** Find the average height of the (single) cone $z = \sqrt{x^2+y^2}$ above the disk $x^2+y^2 \le a^2$ in the xy-plane.

Solution. Average $= \dfrac{1}{\pi a^2}\displaystyle\int_{-a}^{a}\int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \sqrt{x^2+y^2}\,dy\,dx = \dfrac{1}{\pi a^2}\int_0^{2\pi}\int_0^a r^2\,dr\,d\theta$
$= \dfrac{a}{3\pi}\displaystyle\int_0^{2\pi} d\theta = \dfrac{2a}{3}.$ ◀

40. **Volume of noncircular right cylinder** The region enclosed by the lemniscate $r^2 = 2\cos 2\theta$ is the base of a solid right cylinder whose top is bounded by the sphere $z = \sqrt{2-r^2}$. Find the cylinder's volume.

Solution.



Volume $= 2\displaystyle\int_{-\pi/4}^{\pi/4}\int_0^{\sqrt{2\cos(2\theta)}} \sqrt{2-r^2}\cdot r\,dr\,d\theta = 4\int_0^{\pi/4} \frac{-1}{3}\left[(2-2\cos 2\theta)^{3/2} - 2^{3/2}\right] d\theta$
$= \dfrac{2\pi\sqrt{2}}{3} - \dfrac{32}{3}\displaystyle\int_0^{\pi/4}(1-\cos^2\theta)\sin\theta\,d\theta = \frac{2\pi\sqrt{2}}{3} - \frac{32}{3}\left[\frac{\cos^3\theta}{3} - \cos\theta\right]_0^{\pi/4} = \frac{6\pi\sqrt{2}+40\sqrt{2}-64}{9}.$ ◀

42. **Converting to a polar integral** Evaluate the integral

$$\int_0^\infty\int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy.$$

Solution. $\displaystyle\int_0^\infty\int_0^\infty \frac{1}{(1+x^2+y^2)^2}\,dx\,dy = \int_0^{\pi/2}\int_0^\infty \frac{1}{(1+r^2)^2}\cdot r\,dr\,d\theta = \frac{\pi}{2}\lim_{b\to\infty}\int_0^b \frac{r}{(1+r^2)^2}\,dr$
$= \dfrac{\pi}{4}\displaystyle\lim_{b\to\infty}\left[-\frac{1}{(1+r^2)}\right]_0^b = \frac{\pi}{4}\lim_{b\to\infty}\left(1 - \frac{1}{(1+b^2)}\right) = \frac{\pi}{4}.$ ◀

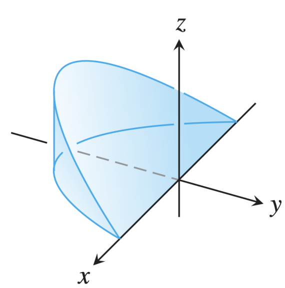
Exercises 15.5

19. Evaluate the integral

$$\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv \quad (tvx\text{-space})$$

Solution. $\int_0^{\pi/4}\int_0^{\ln\sec v}\int_{-\infty}^{2t} e^x\,dx\,dt\,dv = \int_0^{\pi/4}\int_0^{\ln\sec v}\lim_{a\to-\infty}(e^{2t}-e^a)\,dt\,dv$
$= \int_0^{\pi/4}\int_0^{\ln\sec v} e^{2t}\,dt\,dv = \int_0^{\pi/4}\left(\frac{1}{2}e^{2\ln\sec v}-\frac{1}{2}\right)dv = \int_0^{\pi/4}\left(\frac{\sec^2 v}{2}-\frac{1}{2}\right)dv$
$= \left[\frac{\tan v}{2}-\frac{v}{2}\right]_0^{\pi/4} = \frac{1}{2}-\frac{\pi}{8}.$ ◀

26. Find the volume of the region: The wedge cut from the cylinder $x^2+y^2=1$ by the planes $z=-y$ and $z=0$.



Solution. Volume $= \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{0}\int_0^{-y} dz\,dy\,dx = 2\int_0^1\int_{-\sqrt{1-x^2}}^{0} -y\,dy\,dx$
$= \int_0^1 (1-x^2)\,dx = \frac{2}{3}.$ ◀

44. Evaluate the integral by changing the order of integration in an appropriate way.

$$\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx.$$

Solution. $\int_0^2\int_0^{4-x^2}\int_0^x \frac{\sin 2z}{4-z}\,dy\,dz\,dx = \int_0^2\int_0^{4-x^2}\frac{x\sin 2z}{4-z}\,dz\,dx$
$= \int_0^4\int_0^{\sqrt{4-z}}\frac{\sin 2z}{4-z}\cdot x\,dx\,dz = \int_0^4 \frac{\sin 2z}{4-z}\cdot\frac{1}{2}(4-z)\,dz = \left[-\frac{1}{4}\cos 2z\right]_0^4 = \frac{1-\cos 8}{4}$
$= \frac{\sin^2 4}{2}.$ ◀

Exercises 15.7

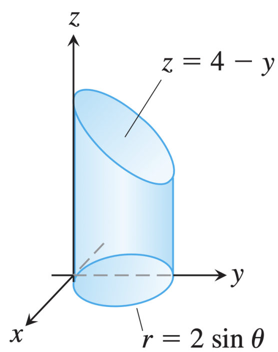
8. Evaluate the integral

$$\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1+\cos\theta} 4r\,dr\,d\theta\,dz.$$

Solution. $\displaystyle\int_{-1}^{1}\int_{0}^{2\pi}\int_{0}^{1+\cos\theta} 4r\,dr\,d\theta\,dz = \int_{-1}^{1}\int_{0}^{2\pi} 2(1+\cos\theta)^2\,d\theta\,dz$

$\displaystyle= \int_{-1}^{1}\int_{0}^{2\pi}(3+\cos 2\theta + 4\cos\theta)\,d\theta\,dz = \int_{-1}^{1} 6\pi\,dz = 12\pi.$ ◀

15. Set up the iterated integral for evaluating $\iiint_D f(r,\theta,z)\,dz\,rdr\,d\theta$ over the given region D. D is the right cylinder whose base is the circle $r = 2\sin\theta$ in the xy-plane and whose top lies in the plane $z = 4 - y$.



Solution. $\displaystyle\int_{0}^{\pi}\int_{0}^{2\sin\theta}\int_{0}^{4-r\sin\theta} f(r,\theta,z)\,dz\,rdr\,d\theta.$ ◀

24. Evaluate the spherical coordinate integral

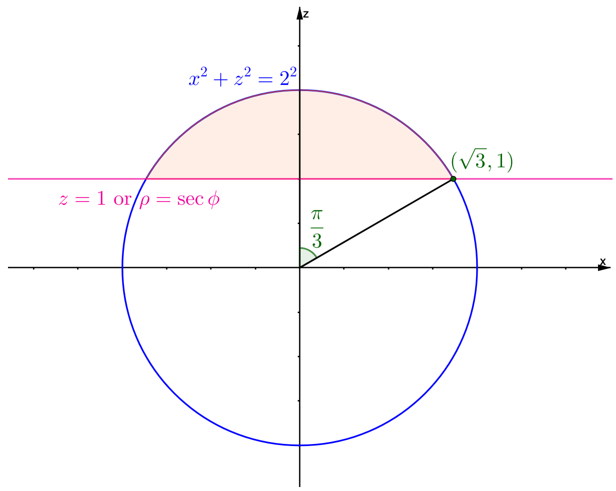
$$\int_{0}^{3\pi/2}\int_{0}^{\pi}\int_{0}^{1} 5\rho^3\sin^3\phi\,d\rho\,d\phi\,d\theta.$$

Solution. $\displaystyle\int_{0}^{3\pi/2}\int_{0}^{\pi}\int_{0}^{1} 5\rho^3\sin^3\phi\,d\rho\,d\phi\,d\theta = \int_{0}^{3\pi/2}\int_{0}^{\pi}\frac{5}{4}\sin^3\phi\,d\phi\,d\theta$

$\displaystyle= \frac{5}{4}\int_{0}^{3\pi/2}\left[\frac{\cos^3\phi}{3} - \cos\phi\right]_{0}^{\pi} d\theta = \frac{5}{4}\int_{0}^{3\pi/2}\frac{4}{3}\,d\theta = \frac{5\pi}{2}.$ ◀

41. Let D be the smaller cap cut from a solid ball of radius 2 units by a plane 1 unit from the center of the sphere. Express the volume of D as an iterated triple integral in **(a)** spherical, **(b)** cylindrical, and **(c)** rectangular coordinates. Then **(d)** find the volume by evaluating one of the three triple integrals.

Solution.



(a) Volume $= \displaystyle\int_0^{2\pi}\int_0^{\pi/3}\int_{\sec\phi}^{2} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta.$

(b) Volume $= \displaystyle\int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta.$

(c) Volume $= \displaystyle\int_{-\sqrt{3}}^{\sqrt{3}}\int_{-\sqrt{3-x^2}}^{\sqrt{3-x^2}}\int_1^{\sqrt{4-x^2-y^2}} dz \, dy \, dx.$

(d) Using cylindrical coordinates,

$$\begin{aligned}\text{Volume} &= \int_0^{2\pi}\int_0^{\sqrt{3}}\int_1^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \left(r\sqrt{4-r^2} - r\right) dr \, d\theta \\ &= \int_0^{2\pi} \left[-\frac{(4-r^2)^{3/2}}{3} - \frac{r^2}{2}\right]_0^{\sqrt{3}} d\theta = \int_0^{2\pi} \left(-\frac{1}{3} - \frac{3}{2} + \frac{4^{3/2}}{3}\right) d\theta \\ &= \int_0^{2\pi} \frac{5}{6} \, d\theta = \frac{5\pi}{3}.\end{aligned}$$

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