MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 2

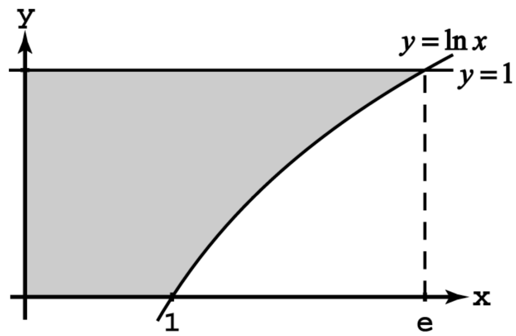
Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercise 15.2

16. Write an iterated integral for $\iint_R dA$ over the described region R using (a) vertical cross-sections, (b) horizontal cross-sections.

    Bounded by $y = 0$, $x = 0$, $y = 1$, and $y = \ln x$.

    Solution.



(a) $\iint_R dA = \int_0^1 \int_0^1 dy\,dx + \int_1^e \int_{\ln x}^1 dy\,dx = e - 1.$

(b) $\iint_R dA = \int_0^1 \int_0^{e^y} dx\,dy = e - 1.$

◀

28. Integrate f over the given region.

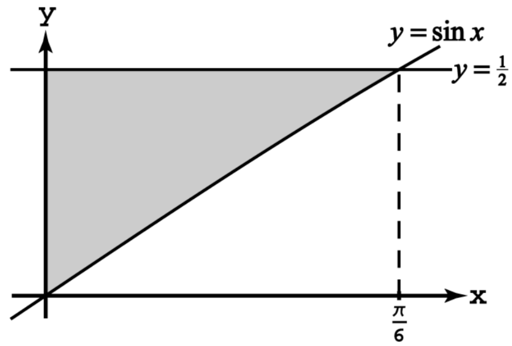
    Curved region $f(s,t) = e^s \ln t$ over the region in the first quadrant of the st-plane that lies above the curve $s = \ln t$ from $t = 1$ to $t = 2$.

Solution. $\int_1^2 \int_0^{\ln t} e^s \ln t\,ds\,dt = \int_1^2 [e^s \ln t]_0^{\ln t}\,dt = \int_1^2 (t\ln t - \ln t)\,dt$

$= \left[\frac{t^2}{2}\ln t - \frac{t^2}{4} - t\ln t + t\right]_1^2 = \frac{1}{4}.$ ◀

44. Sketch the region of integration and write an equivalent double integral with the order of integration reversed. $\int_0^{\pi/6}\int_{\sin x}^{1/2} xy^2\,dy\,dx$

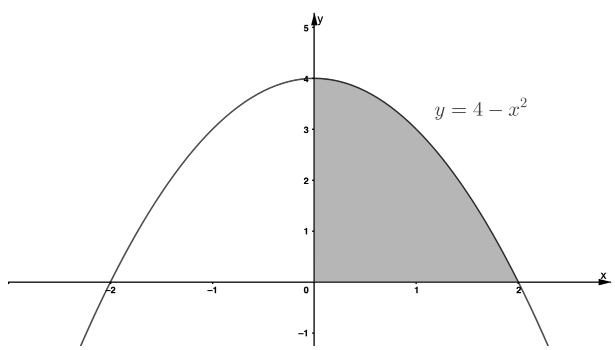
Solution.



$$\int_0^{\pi/6}\int_{\sin x}^{1/2} xy^2\,dy\,dx = \int_0^{1/2}\int_0^{\sin^{-1} y} xy^2\,dx\,dy.$$ ◀

50. Sketch the region of integration, reverse the order of integration, and evaluate the integral. $\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx$

Solution.



$$\int_0^2\int_0^{4-x^2} \frac{xe^{2y}}{4-y}\,dy\,dx = \int_0^4\int_0^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\,dx\,dy = \int_0^4\left[\frac{x^2e^{2y}}{2(4-y)}\right]_0^{\sqrt{4-y}}\,dy = \int_0^4 \frac{e^{2y}}{2}\,dy$$
$$= \left[\frac{e^{2y}}{4}\right]_0^4 = \frac{e^8-1}{4}.$$ ◀

63. Find the volume of the wedge cut from the first octant bounded by the cylinder $z = 12-3y^2$ and the plane $x+y=2$.

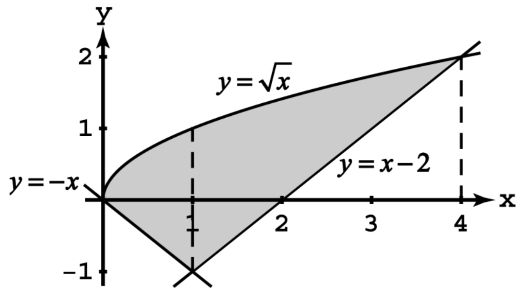
Solution. Volume $= \int_0^2\int_0^{2-x}(12-3y^2)\,dy\,dx = \int_0^2\left[12y-y^3\right]_0^{2-x}\,dx$
$$= \int_0^2\left[12(2-x)-(2-x)^3\right]dx = \left[-6(2-x)^2+\frac{1}{4}(2-x)^4\right]_0^2 = 20.$$ ◀

Exercise 15.3

12. Sketch the region bounded by the given lines and curves. Then express the region's area as an iterated double integral and evaluate the integral.

    The lines $y = x - 2$ and $y = -x$ and the curve $y = \sqrt{x}$.

    Solution.



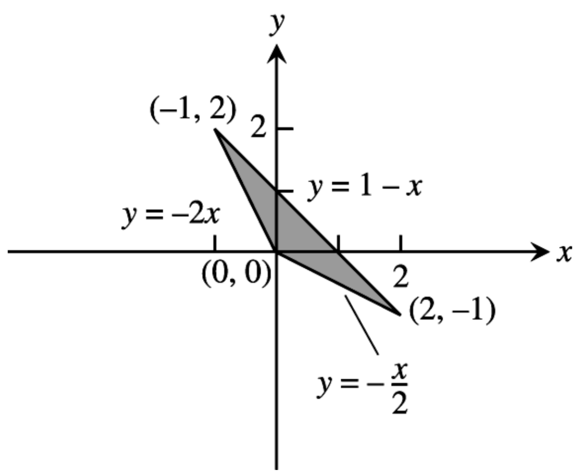
Area $= \int_0^1 \int_{-x}^{\sqrt{x}} dy\,dx + \int_1^4 \int_{x-2}^{\sqrt{x}} dy\,dx = \int_0^1 (\sqrt{x} + x)\,dx + \int_1^4 (\sqrt{x} - x + 2)\,dx$

$= \left[\frac{2}{3}x^{3/2} + \frac{1}{2}x^2\right]_0^1 + \left[\frac{2}{3}x^{3/2} - \frac{1}{2}x^2 + 2x\right]_1^4 = \frac{13}{3}.$ ◀

17. The integrals and sum of integrals give the areas of regions in the xy-plane. Sketch each region, label each bounding curve with its equation, and give the coordinates of the points where the curves intersect. Then find the area of the region.

    $\int_{-1}^0 \int_{-2x}^{1-x} dy\,dx + \int_0^2 \int_{-x/2}^{1-x} dy\,dx$

    Solution.



$\int_{-1}^0 \int_{-2x}^{1-x} dy\,dx + \int_0^2 \int_{-x/2}^{1-x} dy\,dx = \int_{-1}^0 (1 + x)\,dx + \int_0^2 (1 - x/2)\,dx$

$= \left[x + \frac{x^2}{2}\right]_{-1}^0 + \left[x - \frac{x^2}{4}\right]_0^2 = \frac{3}{2}.$ ◀

21. Find the average height of the paraboloid $z = x^2 + y^2$ over the square $0 \le x \le 2$, $0 \le y \le 2$.

Solution. Average $= \dfrac{1}{(2-0)^2}\displaystyle\int_0^2\int_0^2 (x^2+y^2)\,dy\,dx = \frac{1}{4}\int_0^2 \left[x^2y + \frac{y^3}{3}\right]_0^2 dx$
$= \dfrac{1}{4}\displaystyle\int_0^2 \left(2x^2 + \frac{8}{3}\right) dx = \frac{1}{4}\left[\frac{2x^3}{3} + \frac{8x}{3}\right]_0^2 = \frac{8}{3}.$ ◀

Exercise 15.4

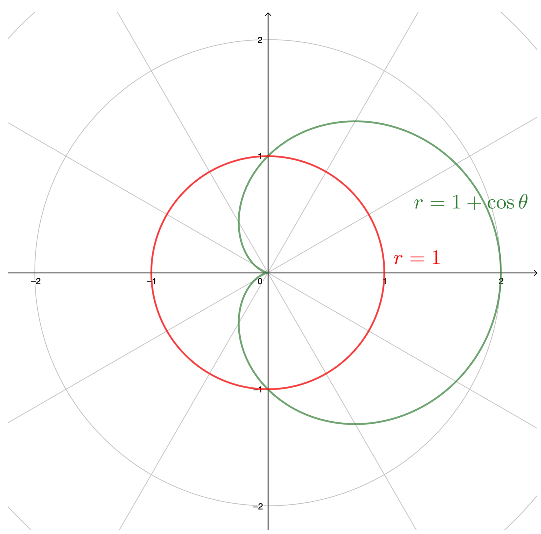
17. Change the Cartesian integral into an equivalent polar integral. Then evaluate the polar integral.

$$\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx$$

Solution. $\displaystyle\int_{-1}^{0}\int_{-\sqrt{1-x^2}}^{0} \frac{2}{1+\sqrt{x^2+y^2}}\,dy\,dx = \int_{\pi}^{3\pi/2}\int_0^1 \frac{2}{1+r}\cdot r\,dr\,d\theta$
$= 2\displaystyle\int_{\pi}^{3\pi/2}\int_0^1 \left(1 - \frac{1}{1+r}\right) dr\,d\theta = 2\int_{\pi}^{3\pi/2} (1-\ln 2)\,d\theta = (1-\ln 2)\pi.$ ◀

28. **Cardioid overlapping a circle** Find the area of the region that lies inside the cardioid $r = 1 + \cos\theta$ and outside the circle $r = 1$.

Solution.



Area $= \displaystyle\int_{-\pi/2}^{\pi/2}\int_1^{1+\cos\theta} r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2}\right]_1^{1+\cos\theta} d\theta$
$= \displaystyle\int_{-\pi/2}^{\pi/2}\left(\cos\theta + \frac{\cos 2\theta + 1}{4}\right) d\theta = \left[\sin\theta + \frac{\sin 2\theta}{8} + \frac{\theta}{4}\right]_{-\pi/2}^{\pi/2} = \frac{8+\pi}{4}.$ ◀