MATH 2020A Advanced Calculus II
2023-24 Term 1
Suggested Solution of Homework 10

Refer to Textbook: Thomas' Calculus, Early Transcendentals, 13th Edition

Exercises 16.7

2. Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

   $\mathbf{F} = 2y\mathbf{i} + 3x\mathbf{j} - z^2\mathbf{k}$

   C: The circle $x^2 + y^2 = 9$ in the xy-plane, counterclockwise when viewed from above.

   Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3x & -z^2 \end{vmatrix} = 0\mathbf{i} - 0\mathbf{j} + (3-2)\mathbf{k} = \mathbf{k},$$

   and C is the boundary of the disk $S \coloneqq \{(x, y, 0) : x^2 + y^2 \leq 9\}$ with upward unit normal $\mathbf{n} = \mathbf{k}$.

   By Stokes' Theorem, the circulation of the field $\mathbf{F}$ around the curve C is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{\{x^2+y^2 \leq 9\}} \mathbf{k} \cdot \mathbf{k}\, dx\, dy = 9\pi.$$

◀

5. Use the surface integral in Stokes' Theorem to calculate the circulation of the field $\mathbf{F}$ around the curve C in the indicated direction.

   $\mathbf{F} = (y^2 + z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} + (x^2 + y^2)\mathbf{k}$

   C: The square bounded by the lines $x = \pm 1$ and $y = \pm 1$ in the xy-plane, counterclockwise when viewed from above.

   Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2 + z^2 & x^2 + y^2 & x^2 + y^2 \end{vmatrix} = 2y\mathbf{i} - (2x - 2z)\mathbf{j} + (2x - 2y)\mathbf{k},$$

   and C is the boundary of the square $S \coloneqq \{(x, y, 0) : |x| \leq 1,\ |y| \leq 1\}$ with upward unit normal $\mathbf{n} = \mathbf{k}$.

   By Stokes' Theorem, the circulation of the field $\mathbf{F}$ around the curve C is

$$\oint_C \mathbf{F} \cdot d\mathbf{r} = \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_{\{|x|,|y| \leq 1\}} (2x - 2y)\, dx\, dy = \int_{-1}^{1} \int_{-1}^{1} (2x - 2y)\, dx\, dy = 0.$$

◀

8. Let $\mathbf{n}$ be the outer unit normal (normal away from the origin) of the parabolic shell

$$S: \quad 4x^2 + y + z^2 = 4, \qquad y \geq 0,$$

and let

$$\mathbf{F} = \left(-z + \frac{1}{2+x}\right)\mathbf{i} + (\tan^{-1} y)\mathbf{j} + \left(x + \frac{1}{4+z}\right)\mathbf{k}.$$

Find the value of

$$\iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma.$$

Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -z + \frac{1}{2+x} & \tan^{-1} y & x + \frac{1}{4+z} \end{vmatrix} = -2\mathbf{j}.$$

The surface S is a level surface given by

$$g(x, y, z) := 4x^2 + y + z^2 = 4, \qquad y \geq 0.$$

Then $\nabla g = 8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k}$ and $\frac{\partial g}{\partial y} = 1 \neq 0$. Hence, an outer unit normal vector of S is $\mathbf{n} = \frac{\nabla g}{|\nabla g|}$, and

$$d\sigma = \frac{|\nabla g|}{\left|\frac{\partial g}{\partial y}\right|}\, dx\, dz = |\nabla g|\, dx\, dz$$

When $y = 0$, $4x^2 + z^2 = 4$. So the projected region on the xz-plane is $\Omega := \{4x^2 + z^2 \leq 4\}$. Therefore,

$$\begin{aligned} \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma &= \iint_\Omega (-2\mathbf{j}) \cdot \frac{1}{|\nabla g|}(8x\mathbf{i} + \mathbf{j} + 2z\mathbf{k})\ |\nabla g|\, dx\, dz \\ &= \iint_\Omega (-2)\, dx\, dz \\ &= -2\text{Area}(\Omega) \\ &= -2\pi \cdot 1 \cdot 2 = -4\pi. \end{aligned}$$

Remark: One may also evaluate the integral by Stokes' Theorem but has to be careful about the orientation of the boundary curve. A parametrization for the curve so that it is oriented anti-clockwisely with respect to the outer normal of the surface is

$$\mathbf{r}(\theta) = (\sin\theta, 0, 2\cos\theta), \qquad 0 \leq \theta \leq 2\pi.$$

◀

15. Use the surface integral in Stokes' Theorem to calculate the flux of the curl of the field $\mathbf{F}$ across the surface S in the direction of the outward unit normal $\mathbf{n}$.

$\mathbf{F} = x^2y\mathbf{i} + 2y^3z\mathbf{j} + 3z\mathbf{k}$

$S: \quad \mathbf{r}(r, \theta) = (r\cos\theta)\mathbf{i} + (r\sin\theta)\mathbf{j} + r\mathbf{k}, \quad 0 \leq r \leq 1, \quad 0 \leq \theta \leq 2\pi.$

Solution. Note that

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 y & 2y^3 z & 3z \end{vmatrix} = -2y^3\mathbf{i} - 0\mathbf{j} - x^2\mathbf{k} = -2y^3\mathbf{i} - x^2\mathbf{k},$$

and

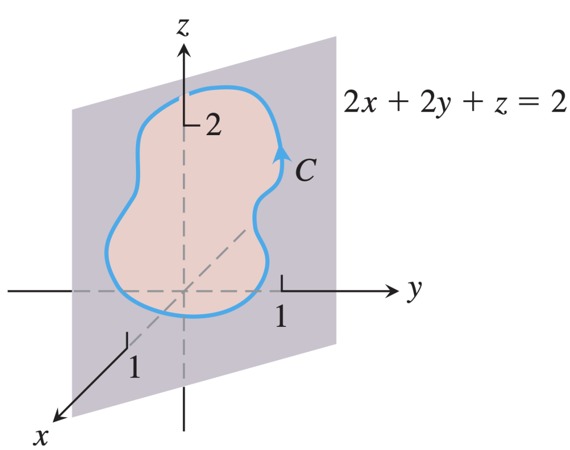
$$\mathbf{r}_r \times \mathbf{r}_\theta = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos\theta & \sin\theta & 1 \\ -r\sin\theta & r\cos\theta & 0 \end{vmatrix} = (-r\cos\theta)\mathbf{i} - (r\sin\theta)\mathbf{j} + r\mathbf{k}.$$

Hence,

$$\begin{aligned} \text{The flux of } \nabla \times \mathbf{F} \text{ across } S &= \iint_S \nabla \times \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iint_R (\nabla \times \mathbf{F}) \cdot (\mathbf{r}_r \times \mathbf{r}_\theta)\, dr\, d\theta \\ &= \int_0^{2\pi} \int_0^1 (2r^4 \sin^3\theta \cos\theta - r^3 \cos^2\theta)\, dr\, d\theta \\ &= \int_0^{2\pi} (-\frac{1}{4}\cos^2\theta)\, d\theta \\ &= -\frac{\pi}{4}. \end{aligned}$$

◀

23. Let C be a simple closed curve in the plane $2x + 2y + z = 2$, oriented as shown here.



Show that

$$\oint_C 2y\, dx - 3z\, dy - x\, dz$$

depends only on the area of the region enclosed by C and not on the position or shape of C

Solution. Let $\mathbf{F} = 2y\mathbf{i} + 3z\mathbf{j} - x\mathbf{k}$ and $g = 2x + 2y + z$. Then

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2y & 3z & -x \end{vmatrix} = -3\mathbf{i} + \mathbf{j} - 2\mathbf{k},$$

and

$$\mathbf{n} = \frac{\nabla g}{|\nabla g|} = \frac{1}{3}(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$$

is a normal vector of the plane such that C is oriented anti-clockwisely with respect to it. By Stokes' Theorem,

$$\begin{aligned}\oint_C 2y\,dx - 3z\,dy - x\,dz = \oint_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \nabla \times \mathbf{F}\,d\sigma \\ &= \iint_S -2\,d\sigma \\ &= -2\iint_S d\sigma,\end{aligned}$$

where $\iint_S d\sigma$ is the area of the region S enclosed by C on the plane $2x + 2y + z = 2$. ◀

Exercises 16.8

6. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

   $\mathbf{F} = x^2\mathbf{i} + y^2\mathbf{j} + z^2\mathbf{k}$

   (a) Cube D: The cube cut from the first octant by the planes $x = 1$, $y = 1$, and $z = 1$

   (b) Cube D: The cube bounded by the planes$x = \pm 1$, $y = \pm 1$, and $z = \pm 1$

   (c) Cylindrical can D: The region cut from the solid cylinder $x^2 + y^2 \leq 4$ by the planes $z = 0$ and $z = 1$.

   Solution. Note that $\nabla \cdot \mathbf{F} = 2x + 2y + 2z$. By Divergence Theorem,

   $$\text{Flux} = \iint_{\partial D} \mathbf{F} \cdot \mathbf{n}\,d\sigma = \iiint_D \nabla \cdot \mathbf{F}\,dV.$$

   (a) Flux $= \int_0^1 \int_0^1 \int_0^1 (2x + 2y + 2z)\,dx\,dy\,dz = 6\int_0^1 \int_0^1 \int_0^1 x\,dx\,dy\,dz = 3.$

   (b) Flux $= \int_{-1}^1 \int_{-1}^1 \int_{-1}^1 (2x + 2y + 2z)\,dx\,dy\,dz = 6\int_{-1}^1 \int_{-1}^1 \int_{-1}^1 x\,dx\,dy\,dz = 0.$

   (c) Using cylindrical coordinates,
   Flux $= \int_0^1 \int_0^{2\pi} \int_0^2 (2r\cos\theta + 2r\sin\theta + 2z)\,r dr\,d\theta\,dz = \int_0^1 \int_0^{2\pi} (\frac{16}{3}\cos\theta + \frac{16}{3}\sin\theta +$ $4z)\,d\theta\,dz = \int_0^1 8\pi z\,dz = 4\pi.$

   ◀

9. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

   Portion of sphere $\mathbf{F} = x^2\mathbf{i} - 2xy\mathbf{j} + 3xz\mathbf{k}$

   D: The region cut from the first octant by the sphere $x^2 + y^2 + z^2 = 4$.

Solution. $\nabla \cdot \mathbf{F} = 2x - 2x + 3x = 3x$. By Divergence Theorem,

$$\text{Flux} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \int_0^{\pi/2} \int_0^{\pi/2} \int_0^2 (3\rho \sin\phi \cos\theta)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$
$$= \int_0^{\pi/2} \int_0^{\pi/2} 12 \sin^2\phi \cos\theta\, d\phi\, d\theta = \int_0^{\pi/2} 3\pi \cos\theta\, d\theta = 3\pi.$$

◀

11. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

Wedge $\quad \mathbf{F} = 2xz\mathbf{i} - xy\mathbf{j} - z^2\mathbf{k}$

D: The wedge cut from the first octant by the plane $y + z = 4$ and the elliptical cylinder $4x^2 + y^2 = 16$.

Solution. $\nabla \cdot \mathbf{F} = 2z - x - 2z = -x$. By Divergence Theorem,

$$\text{Flux} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \int_0^2 \int_0^{\sqrt{16-4x^2}} \int_0^{4-y} -x\, dz\, dy\, dx$$
$$= \int_0^2 \int_0^{\sqrt{16-4x^2}} (xy - 4x)\, dy\, dx = \int_0^2 \left[\frac{1}{2}x(16 - 4x^2) - 4x\sqrt{16 - 4x^2}\right] dx$$
$$= \left[4x^2 - \frac{1}{2}x^4 + \frac{1}{3}(16 - 4x^2)^{3/2}\right]_0^2 = -\frac{40}{3}.$$

◀

14. Use the Divergence Theorem to find the outward flux of $\mathbf{F}$ across the boundary of the region D.

Thick sphere $\quad \mathbf{F} = (x\mathbf{i} + y\mathbf{j} + z\mathbf{k})/\sqrt{x^2 + y^2 + z^2}$

D: The region $1 \le x^2 + y^2 + z^2 \le 4$.

Solution. Let $\rho = \sqrt{x^2 + y^2 + z^2}$. Then $\frac{\partial \rho}{\partial x} = \frac{x}{\rho}$ and $\frac{\partial}{\partial x}(\frac{x}{\rho}) = \frac{1}{\rho} - \left(\frac{x}{\rho^2}\right)\frac{\partial \rho}{\partial x} = \frac{1}{\rho} - \frac{x^2}{\rho^3}$. By symmetry, $\nabla \cdot \mathbf{F} = \frac{3}{\rho} - \frac{x^2+y^2+z^2}{\rho^3} = \frac{2}{\rho}$. By Divergence Theorem,

$$\text{Flux} = \iiint_D \nabla \cdot \mathbf{F}\, dV = \int_0^{2\pi} \int_0^{\pi} \int_1^2 \left(\frac{2}{\rho}\right)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta$$
$$= \int_0^{2\pi} \int_0^{\pi} 3 \sin\phi\, d\phi\, d\theta = \int_0^{2\pi} 6\, d\theta = 12\pi.$$

◀

17. (a) Show that the outward flux of the position vector field $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ through a smooth closed surface S is three times the volume of the region enclosed by the surface.

(b) Let $\mathbf{n}$ be the outward unit normal vector field on S. Show that it is not possible for $\mathbf{F}$ to be orthogonal to $\mathbf{n}$ at every point of S.

Solution. (a) Note that $\nabla \cdot \mathbf{F} = 1 + 1 + 1 = 3$. By Divergence Theorem,

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = \iiint_D \nabla \cdot \mathbf{F}\, dV = 3 \iiint_D dV,$$

where $\iiint_D dV$ is volume of the region enclosed by the surface S.

(b) If $\mathbf{F}$ is orthogonal to $\mathbf{n}$ at every point of S, then $\mathbf{F} \cdot \mathbf{n} = -$ everywhere, and hence

$$\text{Flux} = \iint_S \mathbf{F} \cdot \mathbf{n}\, d\sigma = 0.$$

But Flux $= 3\times$ (volume of D) $\neq 0$, so $\mathbf{F}$ is not orthogonal to $\mathbf{n}$ at every point of S.

◀