eg 1 (in $\mathbb{R}^2$) $\quad \omega = M\,dx + N\,dy \quad (M = M(x,y),\ N = N(x,y))$

then
$$d\omega = dM \wedge dx + dN \wedge dy$$
$$= (M_x\,dx + M_y\,dy) \wedge dx + (N_x\,dx + N_y\,dy) \wedge dy$$
$$= (N_x - M_y)\,dx \wedge dy$$

(+ve) oriented area element

In this notation, Green's Thm $\displaystyle\oint_{C=\partial R} M\,dx + N\,dy = \iint_R (N_x - M_y)\,dx\,dy$,

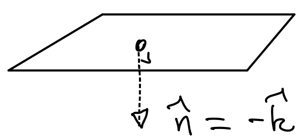
can be written as

$$\boxed{\oint_{C=\partial R} \omega = \iint_R d\omega}$$

Remark: If we let $\vec{F} = M\hat{i} + N\hat{j} \leftrightarrow \omega = M\,dx + N\,dy$

then $(\vec{\nabla} \times \vec{F}) \cdot \hat{n}\,dA = (N_x - M_y)\underbrace{\hat{k} \cdot \hat{n}\,dA}_{dx \wedge dy} = d\omega$

$(\hat{n} = \hat{k})$

and if we use 

then $\hat{k} \cdot \hat{n}\,dA = -dx \wedge dy$

Hence
$$\hat{k} \cdot \hat{n}\,dA = \begin{cases} dx \wedge dy & \text{if } \hat{n} = \hat{k} \\ dy \wedge dx & \text{if } \hat{n} = -\hat{k} \end{cases}$$

(orientation of the "surface")

eg2: $\xi = \xi_1\, dy\wedge dz + \xi_2\, dz\wedge dx + \xi_3\, dx\wedge dy$

Then $d\xi = d\xi_1 \wedge dy\wedge dz + d\xi_2 \wedge dz\wedge dx + d\xi_3 \wedge dx\wedge dy$

$$= \left(\frac{\partial \xi_1}{\partial x} dx + \cdots\right)\wedge dy\wedge dz$$

$$+ \left(\cdots + \frac{\partial \xi_2}{\partial y} dy + \cdots\right)\wedge dz\wedge dx$$

$$+ \left(\cdots + \frac{\partial \xi_3}{\partial z} dz\right)\wedge dx\wedge dy$$

$$= \left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y} + \frac{\partial \xi_3}{\partial z}\right) dx\wedge dy\wedge dz$$

$$= \operatorname{div}\vec{F}\, dx\wedge dy\wedge dz$$

where $\vec{F} = \xi_1\hat{i} + \xi_2\hat{j} + \xi_3\hat{k}$

Hence the divergence theorem can be written as:

$$\iiint_D d\xi = \iiint_D \left(\frac{\partial \xi_1}{\partial x} + \frac{\partial \xi_2}{\partial y} + \frac{\partial \xi_3}{\partial z}\right)\underbrace{dx\wedge dy\wedge dz}$$

(+ve) oriented volume element

$$= \iiint_D \operatorname{div}\vec{F}\, dV = \iint_{S=\partial D} \vec{F}\cdot\hat{n}\, d\sigma$$

outward

To see the relation between $\vec{F}\cdot\hat{n}\,d\sigma$ and ξ,

we parametrize S:

$$\vec{r}(u,v) = x(u,v)\,\hat{i} + y(u,v)\,\hat{j} + z(u,v)\,\hat{k}$$

$$\Rightarrow \begin{cases} \vec{r}_u = x_u\hat{i} + y_u\hat{j} + z_u\hat{k} \\ \vec{r}_v = x_v\hat{i} + y_v\hat{j} + z_v\hat{k} \end{cases}$$

$$\Rightarrow \quad \vec{r}_u \times \vec{r}_v = \begin{vmatrix} y_u & y_v \\ z_u & z_v \end{vmatrix} \hat{i} + \begin{vmatrix} z_u & z_v \\ x_u & x_v \end{vmatrix} \hat{j} + \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} \hat{k}$$

If $\vec{r}_u \times \vec{r}_v$ is <u>outward</u>, then

$$\hat{n} = \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|} \quad \text{and} \quad d\sigma = |\vec{r}_u \times \vec{r}_v| \, du dv$$

(correct orientation)

$$= |\vec{r}_u \times \vec{r}_v| \, du \wedge dv$$

then
$$\vec{F} \cdot \hat{n} \, d\sigma = \vec{F} \cdot \frac{\vec{r}_u \times \vec{r}_v}{\cancel{|\vec{r}_u \times \vec{r}_v|}} \cancel{|\vec{r}_u \times \vec{r}_v|} \, du \wedge dv$$

$$= \left(\zeta_1 \frac{\partial(y,z)}{\partial(u,v)} + \zeta_2 \frac{\partial(z,x)}{\partial(u,v)} + \zeta_3 \frac{\partial(x,y)}{\partial(u,v)} \right) du \wedge dv$$

$$= \zeta_1 \, dy \wedge dz + \zeta_2 \, dz \wedge dx + \zeta_3 \, dx \wedge dy$$

$$= \zeta$$

$$\therefore \iint_S \vec{F} \cdot \hat{n} \, d\sigma = \iint_{(u,v)} \zeta_1 \, dy \wedge dz + \zeta_2 \, dz \wedge dx + \zeta_3 \, dx \wedge dy$$

$$= \iint_{S = \partial D} \zeta$$

Hence divergence thm is

$$\boxed{\iiint_D d\zeta = \iint_{S = \partial D} \zeta}$$

ζ = 2-form

eg3 Stokes' Thm

$$\vec{F} = M\hat{i} + N\hat{j} + L\hat{k} \longleftrightarrow \omega = M\,dx + N\,dy + L\,dz$$

Then

$$\begin{aligned} d\omega = &(L_y - N_z)\,dy\wedge dz \\ &+ (M_z - L_x)\,dz\wedge dx \\ &+ (N_x - M_y)\,dx\wedge dy \end{aligned} \qquad (\text{Ex!})$$

$$= (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma \qquad (\text{Ex!})$$

Stokes' Thm becomes

$$\oint_{C=\partial S} \vec{F}\cdot d\vec{r} \longrightarrow \boxed{\oint_{C=\partial S} \omega = \iint_S d\omega} \longleftarrow \iint_S (\vec{\nabla}\times\vec{F})\cdot\hat{n}\,d\sigma$$

Generalization to manifold of n-dimension with boundary

(Skipped)

- M = n dim'l Manifold (oriented)
- ∂M = boundary (oriented with "induced" orientation)
- $\omega = (n-1)$-form on M (smooth)

Then
$$\boxed{\int_M d\omega = \int_{\partial M} \omega}$$

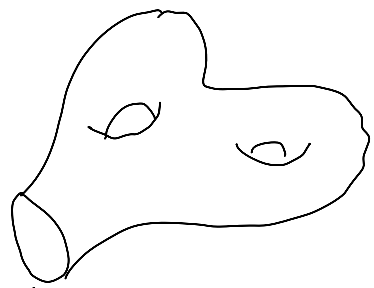
$\int_M$: n-dim'l integral

$\int_{\partial M}$: $(n-1)$-dim'l integral

<u>Note</u>: ∂M is always <u>closed</u>, i.e. no boundary.

$$\therefore \quad \boxed{\partial(\partial M) = \partial^2 M = 0}$$

<u>boundary has no boundary</u>



∂S is a closed curve

Hence if $\omega = d\eta$, for some $(n-2)$-form η,

then $\int_M d(d\eta) = \int_M d\omega = \int_{\partial M} \omega$

$= \int_{\partial M} d\eta = \int_{\partial(\partial M)} \eta = 0$ (for any η.)

This suggests $\boxed{d^2\eta = 0}$, $\forall$ differential form η

<u>Ex</u>: Verify this for 0-form and 1-form in $\mathbb{R}^3$ and observes that these are just

$$\begin{cases} \vec{\nabla} \times \vec{\nabla} f = \vec{0} & (d^2 f = 0) \\ \vec{\nabla} \cdot (\vec{\nabla} \times \vec{F}) = 0 & (d^2\omega = 0) \end{cases}$$

<u>eg</u>: Let $\omega = \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$

<u>check</u>: $d\omega = 0$

But $\omega \neq df$ for any smooth function on $\mathbb{R}^2 \setminus \{(0,0)\}$

(since $\omega = d\theta$ and θ is not defined on $\mathbb{R}^2 \setminus \{(0,0)\}$)

Hence $\quad d\omega = 0 \not\Rightarrow \omega = d\eta \quad$ in general
$(\Leftarrow)$

<u>Note</u>: <u>Thm 10</u> can be written as:

$\boxed{\begin{array}{l} \Omega \subset \mathbb{R}^2 \ \underline{\text{simply-connected}}, \text{ then} \\ \\ d\omega = 0 \Leftrightarrow \omega = df \quad \text{for some smooth function } f \text{ on } \Omega \end{array}}$

Review

Double integrals

- Riemann sum, integrability, Fubini's Thm,
- Polar coordinates, improper integrals
- Applications: area, average, etc.

Triple integrals

- Riemann sum, integrability, Fubini's Thm,
- cylindrical & spherical coordinates, improper integrals
- Applications: volume, average, etc

Change of Variables

- Chain Rule, Jacobian (determinant)

mid-term

Vector Analysis

- Vector fields, gradient of a function ($\vec{\nabla} f$)
- line integral of functions, arc-length
- line integral of vector fields: flow & flux
- simple-closed curves, orientation of curves
- Conservative vector fields (Thms 8, 9 & 10)
- Simply-connected domains
- Curl & Div ($\vec{\nabla} \times \vec{F}$, $\vec{\nabla} \cdot \vec{F}$)

- Surface integrals, area elements, orientation,
- Surface integrals of vector fields (flux)
- Green's, Stokes' & Divergence Thm
- Differential Forms

<u>Final Exam</u> Dec 22 (Fri) 12:30–2:30 U Gym

<u>Coverage</u>:
- All material in lecture notes, tutorial notes, textbook (Ch 15, 16) & homework assignments,
- except differential forms
- emphasis on those material not included in Midterm.
- 6 questions, answer all. Some are unfamiliar/difficult questions as required by the grade descriptor of A range,

(Note: Textbook & assignments contain only basic theory and basic questions.)

(End)