Unified treatment of Green's, Stokes', and Divergence Theorems

Stokes' Thm in notations of differential forms (in $\mathbb{R}^3$)

Working definition of differential forms

(1) A differential 1-form (or simply 1-form)

is a linear combination of the symbols dx, dy & dz:

$$\boxed{\omega = \omega_1\, dx + \omega_2\, dy + \omega_3\, dz}$$

with coefficients $\omega_1, \omega_2, \omega_3$ functions on $\mathbb{R}^3$.

eg: The total differential of a smooth function f

is a differential 1-form:

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

(2) Wedge product: Let "$\wedge$" be an operation such that

$$\begin{cases} dx\wedge dx = dy\wedge dy = dz\wedge dz = 0 \\ dx\wedge dy = -dy\wedge dx \\ dy\wedge dz = -dz\wedge dy \\ dz\wedge dx = -dx\wedge dz \end{cases}$$

and satisfies other usual rules in arithmetic.

i.e. If $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$

$\eta = \eta_1 dx + \eta_2 dy + \eta_3 dz$

then we have

$$\omega \wedge \eta = (\omega_1 dx + \omega_2 dy + \omega_3 dz) \wedge (\eta_1 dx + \eta_2 dy + \eta_3 dz)$$

$$= \omega_1 \cancel{dx \wedge \eta_1 dx}^{0} + \omega_2 dy \wedge \eta_1 dx + \omega_3 dz \wedge \eta_1 dx$$

$$+ \omega_1 dx \wedge \eta_2 dy + \omega_2 \cancel{dy \wedge \eta_2 dy}^{0} + \omega_3 dz \wedge \eta_2 dy$$

$$+ \omega_1 dx \wedge \eta_3 dz + \omega_2 dy \wedge \eta_3 dz + \omega_3 \cancel{dz \wedge \eta_3 dz}^{0}$$

$$= (\omega_1 \eta_2 - \omega_2 \eta_1)\, dx \wedge dy$$

$$+ (\omega_2 \eta_3 - \omega_3 \eta_2)\, dy \wedge dz$$

$$+ (\omega_3 \eta_1 - \omega_1 \eta_3)\, dz \wedge dx$$

$$\therefore \boxed{\begin{aligned} \omega \wedge \eta = (\omega_2 \eta_3 - \omega_3 \eta_2)\, dy \wedge dz \\ + (\omega_3 \eta_1 - \omega_1 \eta_3)\, dz \wedge dx \\ + (\omega_1 \eta_2 - \omega_2 \eta_1)\, dx \wedge dy \end{aligned}}$$

- Linear combinations of $dy \wedge dz$, $dz \wedge dx$ & $dx \wedge dy$ are called <u>differential 2-forms</u> (on $\mathbb{R}^3$)

$$\boxed{\zeta = \zeta_1\, dy \wedge dz + \zeta_2\, dz \wedge dx + \zeta_3\, dx \wedge dy}$$

Similarly, if ω is a 1-form and ζ is a 2-form

then we can define $\omega \wedge \zeta$

eg: If $\omega = dx$, $\quad \zeta = dy \wedge dz$

then $\omega \wedge \zeta = dx \wedge dy \wedge dz$

Note that we insist on the anti-commutativity of wedge product, we have

$$\begin{aligned} dx \wedge dy \wedge dz &= -dy \wedge dx \wedge dz \\ &= dy \wedge dz \wedge dx \\ &= -dz \wedge dy \wedge dx \\ &= dz \wedge dx \wedge dy \\ &= -dx \wedge dz \wedge dy \end{aligned}$$

And $\quad dx \wedge dx \wedge dy = \cdots = 0 \quad$ whenever one of the dx, dy, dz repeated.

Hence, as $\dim \mathbb{R}^3 = 3$, all "linear combinations" of "3-forms" are just $\quad f\, dx \wedge dy \wedge dz$

which is called a differential 3-form (also called a volume form if $f > 0$)

Note: It is convenient to call smooth functions f the differential 0-form.

Summary (on $\mathbb{R}^3$)

0-form	=	f
1-form	=	$\omega_1 dx + \omega_2 dy + \omega_3 dz$
2-form	=	$\xi_1 dy\wedge dz + \xi_2 dz\wedge dx + \xi_3 dx\wedge dy$
3-form	=	$g\, dx\wedge dy\wedge dz$

where, f, g, ω_i, ξ_i are (smooth) functions

Note: One can certainly define k-form for any $k\geq 0$. But in $\mathbb{R}^3$, k-forms are zero for $k>3$:

$dx^i \wedge dx\wedge dy\wedge dz = 0$, where $dx^i = dx, dy,$ or dz.

Change of Variables Formula: ($\mathbb{R}^2$)

$$\begin{cases} x = x(u,v) \\ y = y(u,v) \end{cases}$$

$$\Rightarrow \begin{cases} dx = x_u du + x_v dv \\ dy = y_u du + y_v dv \end{cases}$$

$$\Rightarrow dx\wedge dy = (x_u du + x_v dv)\wedge(y_u du + y_v dv)$$

$$= (x_u y_v - x_v y_u)\, du\wedge dv$$

$$= \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} du\wedge dv$$

$$\boxed{dx\wedge dy = \frac{\partial(x,y)}{\partial(u,v)}\, du\wedge dv}$$

Jacobian determinant.

Hence naturally

$$\boxed{\iint f(x,y)\, dx\wedge dy = \iint f(x(u,v),y(u,v))\frac{\partial(x,y)}{\partial(u,v)}\, du\wedge dv}$$

<u>Compare</u> with

$$\boxed{\iint f(x,y)\, dx\, dy = \iint f(x(u,v),y(u,v))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\, dv}$$

Similarly for $\begin{cases} x = x(u,v,w) \\ y = y(u,v,w) \\ z = z(u,v,w) \end{cases}$

$$\boxed{dx\wedge dy\wedge dz = \frac{\partial(x,y,z)}{\partial(u,v,w)}\, du\wedge dv\wedge dw} \qquad (\text{Ex!})$$

(using $dx = x_u du + x_v dv + x_w dw, \cdots$)

- "Oriented" change of variables formula
- "$dx\wedge dy$" oriented area element
- "$dx\wedge dy\wedge dz$" oriented volume element.

(see later remark)

Exterior differentiation "d" on a form "ω".

(0-form) f	df (1-form)
(1-form) $\omega = \omega_1 dx + \omega_2 dy + \omega_3 dz$	$d\omega = d\omega_1 \wedge dx + d\omega_2 \wedge dy + d\omega_3 \wedge dz$ (2-form)
$\xi = \xi_1 dy \wedge dz + \xi_2 dz \wedge dx + \xi_3 dx \wedge dy$ (2-form)	$d\xi = d\xi_1 \wedge dy \wedge dz + d\xi_2 \wedge dz \wedge dx + d\xi_3 \wedge dx \wedge dy$ (3-form)
(3-form) $f\, dx \wedge dy \wedge dz$	$df \wedge dx \wedge dy \wedge dz = 0$ (4-form) in $\mathbb{R}^3$

eg 0 $d(dx) = d(dy) = d(dz) = 0$. $(d^2x = d^2y = d^2z = 0)$